

CELLULAR LEGENDRIAN CONTACT HOMOLOGY FOR SURFACES, PART I

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ABSTRACT. We give a computation of the Legendrian contact homology (LCH) DGA for an arbitrary generic Legendrian surface L in the 1-jet space of a surface. As input we require a suitable cellular decomposition of the base projection of L . A collection of generators is associated to each cell, and the differential is given by explicit matrix formulas. In the present article, we prove that the equivalence class of this cellular DGA does not depend on the choice of decomposition, and in the sequel [35] we use this result to show that the cellular DGA is equivalent to the usual Legendrian contact homology DGA defined via holomorphic curves. Extensions are made to allow Legendrians with non-generic cone-point singularities. We apply our approach to compute the LCH DGA for several examples including an infinite family, and to give general formulas for DGAs of front spinings allowing for the axis of symmetry to intersect L .

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1. INTRODUCTION

The Legendrian contact homology (abbrv. LCH) algebra of a Legendrian submanifold, L , in the 1-jet space, J^1M , of a manifold M is a differential graded algebra (abbrv. DGA) with generators determined by the double points of the Lagrangian projection of L and with differential defined via counts of suitable spaces of pseudo-holomorphic disks. The LCH algebra was outlined in a more general setting as part of the wider class of symplectic field theory invariants in [22] and, for Legendrian submanifolds in 1-jet spaces, was first rigorously constructed in [14, 17].

Aside from being able to distinguish Legendrian isotopy classes, the LCH algebra is useful in identifying the border between symplectic flexibility and rigidity. For example, the existence of an augmentation of the LCH algebra of a displaceable Legendrian $L \subset J^1M$ implies that L satisfies the Arnold conjecture for exact Lagrangian immersions—roughly speaking, the projection of L to T^*M must have enough double points to generate half of the homology of L [16, 12]. In contrast, Legendrian submanifolds whose LCH algebras have vanishing homology have been constructed to have only 1 double point [11]. A sample of other applications of LCH algebras include (i) surgery exact triangles explaining the effect of Legendrian surgery/symplectic handle attachment on the symplectic and contact homology of Stein manifolds and their contact boundaries [2]; (ii) a complete invariant for smooth knots in \mathbb{R}^3 [32, 13, 25, 20] (iii) connections between the LCH algebra of 1-dimensional Legendrian knots and topological invariants of knots such as the Kauffman and HOMFLY polynomial [34, 29]; (iv) results concerning Lagrangian cobordisms in symplectizations [18, 8, 4].

For 1-dimensional Legendrians (in a 1-jet space), the LCH algebra is combinatorially computable, as the pseudo-holomorphic disks used to define the differential may be identified using the Riemann mapping theorem [5]. Moreover, after placing the front diagram of L into a standard form, the count of such disks can be carried out in an algorithmic manner [31]. As a result, the LCH algebra is an effective

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tool for distinguishing Legendrian knots with reasonably sized diagrams, and is useful in distinguishing many pairs of Legendrian knots in the current tables.

In contrast, the explicit computation of Legendrian contact homology for higher dimensional Legendrians is much more challenging as direct counts of the relevant moduli of pseudo-holomorphic disks are not feasible. For Legendrians in 1-jet spaces, a major step towards explicit computation was carried out in [10] where it was shown that the count of pseudo-holomorphic disks can be replaced by a count of certain rigid gradient flow trees (abbrv. GFTs). This replaces the need to identify pseudo-holomorphic disks with a problem phrased entirely in terms of ODEs. However, GFTs are themselves complicated global objects. Consequently, explicit computations of the full DGA have been limited to several restricted classes of 2-dimensional Legendrians: conormal lifts of smooth knots in \mathbb{R}^3 [13]; two-sheeted Legendrian surfaces [7]; and Legendrian tori constructed from 1-dimensional Legendrian knots via front and isotopy spinnings [19]. There have also been a number of partial DGA computations in higher dimensions: front spinning [15, 9]; and DGA maps induced from Lagrangian cobordisms [8, 30].

In this paper and its companion [35], we give a computation of the LCH algebra with $\mathbb{Z}/2$ -coefficients for all 2-dimensional Legendrians in the 1-jet space of a surface. For a surface S and a Legendrian $L \subset J^1S$, we define a cellular DGA, (\mathcal{A}, ∂) , that requires as input a suitable polygonal decomposition of the base projection (to S) of L . To each 0-, 1-, and 2-cell in the decomposition we associate a collection of generators with the precise number of generators determined by the appearance of the front projection of L above the cell. After placing the generators associated to a particular cell and its boundary cells into matrices, the differential of (\mathcal{A}, ∂) is characterized by explicit formulas. See Figure 4 for a brief summary.

The standard notion of equivalence used when considering Legendrian contact homology DGAs is known as stable tame isomorphism. This is at least as strong as the notion of homotopy equivalence for DGAs. Our main result is:

Theorem 1.1. *For any 2-dimensional $L \subset J^1S$, the cellular DGA (\mathcal{A}, ∂) is equivalent to the Legendrian contact homology DGA of L with coefficients in $\mathbb{Z}/2$. In particular, up to stable tame isomorphism, (\mathcal{A}, ∂) does not depend on the choice of polygonal decomposition, and (\mathcal{A}, ∂) is a Legendrian isotopy invariant.*

In the present article, we define the cellular DGA, and prove independence of the choice of polygonal decomposition. In [35], we construct a stable tame isomorphism between the Legendrian contact homology DGA of L and the cellular DGA of L obtained from a particular choice of polygonal decomposition. Here, we use the reformulation of the differential in terms of GFTs [10]. We produce relatively explicit coordinate models for Legendrian surfaces to make our computations of LCH.

The differential in the cellular DGA is local in the sense that it preserves sub-algebras corresponding to generators associated to the closure of any particular cell. Starting with the splash construction [23], similar localization techniques have been developed in the case of 1-dimensional Legendrians where they have been applied to study augmentations of the LCH algebra and to relate the LCH algebra with invariants defined via the techniques of generating families and constructible sheaves [24, 38, 33]. The localizing splash construction has since been generalized to higher dimensional Legendrians [26, 30]. We expect that the cellular DGA may also provide an approach for extending the other applications to the 2-dimensional case. In particular, in the upcoming article [36] we apply Theorem 1.1 to give a necessary and sufficient condition for the existence of augmentations in terms of a 2-dimensional generalization of the Morse Complex Sequences studied in [28], which exists in the presence of a generating family for the Legendrian.

Although the number of generators of (\mathcal{A}, ∂) can be quite large, it is often possible to reduce the number of generators to a much smaller size while retaining an explicit formula for the differential. The idea is to cancel generators in pairs via a suitable stable tame isomorphisms. Throughout Section 5, we illustrate this process with explicit examples. In that section we also show how to relax conditions on the cell decomposition (for example, allowing cone points which arise naturally in [13, 7]), which further reduces the set of generators. See Sections 5.3 and 5.4.

In Section 2, we recall background material on DGAs and Legendrian surfaces in 1-jet spaces. The definition of the cellular DGA is given in Section 3, where it is verified that differential ∂ has degree -1 and satisfies $\partial^2 = 0$; see Theorem 3.2. Theorem 4.1 of Section 4 shows that for any fixed Legendrian

$L \subset J^1 S$, the stable tame isomorphism type of the cellular DGA is independent of the choice of polygonal decomposition and of other auxiliary choices. In Section 5.1 and 5.2, we compute the LCH of two Legendrian spheres. In Section 5.5 we compute the LCH for front-spun knots, expressing it as a natural DGA operation. Some of these computations agree with pre-existing ones. In Section 5.6, we provide a family of spheres many of which have identical Linearized homology groups, but are distinguished by product operations that would be difficult to identify without explicit formulas for differentials.

We remark that knowledge of LCH is neither assumed nor required in the current article. The required background on LCH and GFTs is postponed until [35].

2. BACKGROUND

2.1. Differential graded algebras. In this article, we consider Differential Graded Algebras (abbrv. **DGAs**) with $\mathbb{Z}/2$ -coefficients and \mathbb{Z}/m -grading for some non-negative, integer m . The differential $\partial : \mathcal{A} \rightarrow \mathcal{A}$ of a DGA (\mathcal{A}, ∂) has degree -1 and satisfies the Liebniz rule $\partial(x \cdot y) = (\partial x)y + (-1)^{|x|}x(\partial y)$.

A **based DGA** is a DGA (\mathcal{A}, ∂) with a chosen subset $\mathcal{B} = \{q_1, \dots, q_r\} \subset \mathcal{A}$ such that \mathcal{A} is freely generated by \mathcal{B} , as an associative (non-commutative) $\mathbb{Z}/2$ -algebra with identity. That is, $\mathcal{A} \cong (\mathbb{Z}/2)\langle q_1, \dots, q_r \rangle$ so that elements of \mathcal{A} can be written uniquely as $\mathbb{Z}/2$ -linear combinations of words in the elements of \mathcal{B} . Furthermore, we require that the generating set \mathcal{B} consists of homogeneous elements, so that the \mathbb{Z}/m -grading, $\mathcal{A} = \bigoplus_{i \in \mathbb{Z}/m} \mathcal{A}_i$, is determined by the assignment of degrees to generators, $|\cdot| : \mathcal{B} \rightarrow \mathbb{Z}/m$, and the requirement that $|x \cdot y| = |x| + |y|$ when x and y are homogeneous elements.

We say that the differential ∂ is **triangular** with respect to a total ordering of the generating set given by $\mathcal{B} = \{q_1, \dots, q_n\}$ if for all $1 \leq i \leq n$, $\partial q_i \in F_{i-1}\mathcal{A}$ where $F_0\mathcal{A} = \mathbb{Z}/2$ and $F_i\mathcal{A} \subset \mathcal{A}$ denotes the subalgebra generated by q_1, \dots, q_i .

2.1.1. Stable tame isomorphism. An **elementary automorphism** of \mathcal{A} is a graded algebra map $\phi : \mathcal{A} \rightarrow \mathcal{A}$ such that for some $q_i \in \mathcal{B}$, we have $\phi(q_j) = q_j$ when $j \neq i$, and $\phi(q_i) = q_i + v$ where v belongs to the subalgebra generated by $\mathcal{B} \setminus \{q_i\}$. A **tame isomorphism** between based DGAs is an isomorphism of DGAs $\psi : (\mathcal{A}, \partial) \rightarrow (\mathcal{A}', \partial')$, i.e. a graded algebra isomorphism satisfying $\psi \circ \partial = \partial' \circ \psi$, that is a composition of elementary automorphisms of \mathcal{A} followed by an isomorphism $\mathcal{A} \rightarrow \mathcal{A}'$ that extends some bijection of generating sets $\mathcal{B} \cong \mathcal{B}'$.

The **degree i stabilization** of a based DGA, (\mathcal{A}, ∂) , with generating set, \mathcal{B} , is the based DGA, $(S\mathcal{A}, \partial')$, with generating set, $S\mathcal{B} = \mathcal{B} \cup \{a, b\}$, where the new generators are assigned degrees $|a| = i$, $|b| = i - 1$, and ∂' satisfies

$$\partial'a = b; \quad \partial'b = 0; \quad \text{and} \quad \partial'|_{\mathcal{A}} = \partial.$$

Definition 2.1. *Two based DGAs are **stable tame isomorphic** if after stabilizing each of the DGAs some (possibly different) number of times they become isomorphic by a tame isomorphism.*

It can be checked that stable tame isomorphism satisfies the properties of an equivalence relation. We occasionally say that two based DGAs are **equivalent** to mean that they are stable tame isomorphic.

The following theorem will serve as our primary method for producing stable tame isomorphisms.

Theorem 2.1. *Let (\mathcal{A}, ∂) be a based DGA such that ∂ is triangular with respect to the ordered generating set $\mathcal{B} = \{q_1, \dots, q_n\}$. Suppose that*

$$\partial q_k = q_l + w \text{ where } w \in F_{l-1}\mathcal{A},$$

and let $I = I(q_k, \partial q_k)$ denote the 2-sided ideal generated by q_k and ∂q_k . Then, $(\mathcal{A}/I, \partial)$ is free and triangular with respect to the ordered generating set $\mathcal{B}' = \{[q_1], \dots, [\widehat{q_l}], \dots, [\widehat{q_k}], \dots, [q_n]\}$. Moreover, with these generating sets, (\mathcal{A}, ∂) and $(\mathcal{A}/I, \partial)$ are stable tame isomorphic.

Typically we use the same notation for an element of \mathcal{A} and its equivalence class in a quotient of \mathcal{A} . Note that to write the differential ∂q_i in \mathcal{A}/I in terms of the generating set, \mathcal{B}' , we simply replace all occurrences of q_k (resp. q_l) in $\partial q_i \in \mathcal{A}$ by 0 (resp. by w).

Proof. This is essentially due to [5, Section 8.4].

If necessary, we can reorder $\{q_1, \dots, q_n\}$ as $\{q_1, \dots, q_l, q_k, q_{l+1}, \dots, \widehat{q_k}, \dots, q_n\}$ to arrange that k and l are consecutive while preserving the triangular property of ∂ . Thus, we can suppose $\partial q_k = q_{k-1} + w$

with $w \in F_{k-2}\mathcal{A}$, and show $(\mathcal{A}/I, \partial)$ is stable tame isomorphic to (\mathcal{A}, ∂) where $I := I(q_k, \partial q_k)$. It is clear that \mathcal{A}/I is indeed freely generated by \mathcal{B}' since we have the vector space decomposition, $\mathcal{A} = \mathcal{A}' \oplus I$ where $\mathcal{A}' \subset \mathcal{A}$ is the subalgebra generated by $\mathcal{B} \setminus \{q_k, q_{k-1}\}$. We next need the following.

Lemma 2.1. *There exists a DGA map $f : \mathcal{A}/I \rightarrow \mathcal{A}$ such that for any $[q_i] \in \mathcal{B}'$, $(i \neq k, k-1)$, $f([q_i]) = q_i + w_i$ with $w_i \in F_{i-1}\mathcal{A}$.*

Proof. The projection $p : \mathcal{A} \rightarrow \mathcal{A}/I$ is a DGA map. We will construct f to be a homotopy inverse to p .

Note that $(\mathcal{A}/I, \partial)$ inherits a filtration from (\mathcal{A}, ∂) , where $F_i(\mathcal{A}/I) = p(F_i\mathcal{A})$ is the sub-algebra generated by $[q_1], \dots, [q_i]$. In particular, $F_k(\mathcal{A}/I) = F_{k-1}(\mathcal{A}/I) = F_{k-2}(\mathcal{A}/I)$ are all freely generated by $[q_i]$ with $1 \leq i \leq k-2$. Begin by defining $f_0 : F_k\mathcal{A}/I \rightarrow F_k\mathcal{A}$, and $H_0 : F_k\mathcal{A} \rightarrow F_k\mathcal{A}$ on generators by

$$f_0([q_i]) = q_i, \quad \text{for } 1 \leq i \leq k-2; \quad H_0(q_i) = \begin{cases} q_k, & \text{if } i = k-1, \\ 0 & \text{else.} \end{cases}$$

Then, extend f_0 as an algebra homomorphism, and extend H_0 to be a $(f_0 \circ p, id_{F_k\mathcal{A}})$ -derivation, meaning that $H_0(x \cdot y) = H_0(x) \cdot id_{\mathcal{A}_0}(y) + (f_0 \circ p)(x) \cdot H_0(y)$. A direct verification shows that

$$f_0 \circ \partial_{\mathcal{A}/I} = \partial_{\mathcal{A}} \circ f_0; \quad \text{and} \quad f_0 \circ p - id_{\mathcal{A}_0} = \partial_{\mathcal{A}} \circ H_0 + H_0 \circ \partial_{\mathcal{A}}$$

hold when applied to generators of $F_k(\mathcal{A}/I)$ and $F_k\mathcal{A}$, respectively. This implies that the identities in fact hold on all of $F_k(\mathcal{A}/I)$ and $F_k\mathcal{A}$. [In general, if $g_1, g_2 : (\mathcal{A}_1, \partial_1) \rightarrow (\mathcal{A}_2, \partial_2)$ are DGA maps, and $H : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ is a (g_1, g_2) -derivation, then $g_1 - g_2 = \partial_{\mathcal{A}_2} \circ H + H \circ \partial_{\mathcal{A}_1}$ holds if and only if it holds on a generating set for \mathcal{A}_1 .]

Now, for $i \geq 0$, we inductively define maps $f_i : (F_{k+i}\mathcal{A}/I, \partial) \rightarrow (F_{k+i}\mathcal{A}, \partial)$ and $H_i : (F_{k+i}\mathcal{A}, \partial) \rightarrow (F_{k+i}\mathcal{A}, \partial)$ to satisfy

- (1) f_i is a chain map,
- (2) H_i is a $(f_i \circ p, id_{F_{k+i}\mathcal{A}})$ -derivation, and
- (3) $f_i \circ p - id_{F_{k+i}\mathcal{A}} = \partial_{\mathcal{A}} \circ H_i + H_i \circ \partial_{\mathcal{A}}$.

Supposing f_i and H_i have been defined, we extend them to maps $f_{i+1} : (F_{k+i+1}\mathcal{A}/I, \partial) \rightarrow (F_{k+i+1}\mathcal{A}, \partial)$ and $H_{i+1} : (F_{k+i+1}\mathcal{A}, \partial) \rightarrow (F_{k+i+1}\mathcal{A}, \partial)$. Begin by setting

$$f_{i+1}([q_{i+1}]) = q_{i+1} + H_i(\partial_{\mathcal{A}}(q_{i+1})).$$

Note that $\partial_{\mathcal{A}}(q_{i+1}) \in F_i\mathcal{A}$, since $\partial_{\mathcal{A}}$ is triangular. Thus, this definition makes sense, and we indeed have $f_{i+1}([q_{i+1}]) = q_{i+1} + w_{i+1}$ with $w_{i+1} \in F_i\mathcal{A}$. Next, extend f_{i+1} as an algebra homomorphism, put $H_{i+1}(q_{i+1}) = 0$, and then extend H_{i+1} to $F_{k+i+1}\mathcal{A}$ as a $(f_{i+1} \circ p, id_{F_{k+i+1}\mathcal{A}})$ -derivation. We check that f_{i+1} and H_{i+1} again satisfy (1)-(3). Note that (2) holds by construction. Once (1) is known, (3) follows from the definition of $f_{i+1}([q_{i+1}])$ since the identity holds when applied to generators of $F_{k+i+1}\mathcal{A}$.

To verify (1), compute:

$$\begin{aligned} f_{i+1} \circ \partial_{\mathcal{A}/I}([q_{i+1}]) &= f_{i+1} \circ p(\partial_{\mathcal{A}}q_{i+1}) = f_i \circ p(\partial_{\mathcal{A}}q_{i+1}) = \partial_{\mathcal{A}}q_{i+1} + (\partial_{\mathcal{A}} \circ H_i + H_i \circ \partial_{\mathcal{A}})(\partial_{\mathcal{A}}q_{i+1}) = \\ &= \partial_{\mathcal{A}}q_{i+1} + \partial_{\mathcal{A}} \circ H_i(\partial_{\mathcal{A}}q_{i+1}) = \partial_{\mathcal{A}}(q_{i+1} + H_i(\partial_{\mathcal{A}}q_{i+1})) = \partial_{\mathcal{A}} \circ f_{i+1}([q_{i+1}]) \end{aligned}$$

where at the second equality we used that $\partial_{\mathcal{A}}q_{i+1} \in F_i\mathcal{A}$ since ∂ is triangular.

Thus, the construction proceeds inductively with $f = f_n$ satisfying the stated requirements of the Lemma. \square

We now construct a stable tame isomorphism. Denote the generators of the stabilization, $(S(\mathcal{A}/I), \partial')$, as $\bar{q}_1, \dots, \bar{q}_n$ where, for $i \notin \{k, k-1\}$, $\bar{q}_i = [q_i]$, and \bar{q}_k and \bar{q}_{k-1} are the two new generators from stabilization with $\partial'\bar{q}_k = \bar{q}_{k-1}$; the degree of the stabilization is chosen so that $|\bar{q}_k| = |q_k|$ and $|\bar{q}_{k-1}| = |q_{k-1}|$. Define an algebra map $F : (S(\mathcal{A}/I), \partial') \rightarrow (\mathcal{A}, \partial)$ by

$$F(\bar{q}_i) = \begin{cases} f(\bar{q}_i), & \text{for } i \notin \{k, k-1\}, \\ q_k, & \text{for } i = k, \\ q_{k-1} + w, & \text{for } i = k-1. \end{cases}$$

F is a chain map: It suffices to verify that $F \circ \partial' = \partial \circ F$ holds when applied to generators. Compute

$$\begin{aligned} F \circ \partial'(\bar{q}_k) &= F(\bar{q}_{k-1}) = q_{k-1} + w = \partial q_k = \partial \circ F(\bar{q}_k); \\ F \circ \partial'(\bar{q}_{k-1}) &= 0 = \partial^2 q_k = \partial \circ F(\bar{q}_{k-1}). \end{aligned}$$

For $i \notin \{k, k-1\}$, $\bar{q}_i \in \mathcal{A}/I$, so we have

$$(F \circ \partial')(\bar{q}_i) = f(\partial_{\mathcal{A}/I}([q_i])) = \partial_A(f([q_i])) = (\partial \circ F)(\bar{q}_i).$$

F is a tame isomorphism: Note that for all $1 \leq i \leq n$, $F(\bar{q}_i) = q_i + w_i$ where $w_i \in F_{i-1}\mathcal{A}$. Thus, we can write

$$F = f_n \circ \cdots \circ f_1 \circ \sigma$$

$$\text{where } f_i(q_j) = \begin{cases} q_i + w_i & \text{for } i = j \\ q_j & \text{for } j \neq i, \end{cases} \text{ and } \sigma(\bar{q}_i) = q_i.$$

□

Remark 2.2. (i) The notion of stable tame isomorphism appears in [5] and is now prevalent in the literature surrounding Legendrian contact homology.

(ii) A variation on stable tame isomorphism arises from requiring that the word w that appears in $\phi(q_i)$ in the definition of elementary isomorphism belongs to $F_{i-1}\mathcal{A}$ with respect to some ordering of generators for which ∂ is triangular. We do not know whether the equivalence on based algebras generated by such isomorphisms is distinct from stable tame isomorphism. However, we note in passing that the proof of Theorem 4.1 shows that the DGAs corresponding to distinct compatible cell decompositions of L are related by this stronger equivalence.

2.2. Legendrian surfaces. Let S be a surface. The 1-jet space, $J^1S = T^*S \times \mathbb{R}$, has a standard contact structure given by the kernel of the contact form $dz - \lambda$ where λ is the Liouville 1-form on T^*S and z is the coordinate on the \mathbb{R} factor. Any choice of local coordinates (x_1, x_2) on $U \subset S$ leads to coordinates (x_1, x_2, y_1, y_2, z) on $J^1U \subset J^1S$ with respect to which the contact form appears as

$$dz - \lambda = dz - \sum_{i=1}^2 y_i dx_i.$$

We denote by

$$\pi_{xz} : J^1S \rightarrow S \times \mathbb{R}; \quad \text{and} \quad \pi_x : J^1S \rightarrow S$$

the projections which are known respectively as the **front projection** and **base projection**.

A surface $L \subset J^1S$ is **Legendrian** if $(dz - \lambda)|_L = 0$. The singularities of front and base projections of Legendrian submanifolds are well studied, cf. [1]. In this 2-dimensional setting, after possibly perturbing L by a small Legendrian isotopy, we may assume that $L = \bigsqcup_{i=0}^2 L_i$ such that $L_1 \sqcup L_2 \subset L$ is an embedded 1-manifold and L_2 is a discrete set; near each L_i the front projection has the following standard forms.

- Any $p \in L_0$ has a neighborhood, $N \subset L$, whose front projection is the graph of a smooth function defined on a neighborhood U of $\pi_x(p)$, i.e.

$$\exists f : U \rightarrow \mathbb{R}, \quad \text{such that} \quad \pi_{xz}(N) = \{(x, z) \mid z = f(x)\}.$$

In particular, the restriction of π_{xz} to L_0 is an immersion. We often refer to connected subsets of L_0 and/or their front projections as **sheets** of L .

- Any point in L_1 has a neighborhood whose front projection is diffeomorphic to a standard **cusp edge**, i.e. a semi-cubical cusp crossed with an interval, see Figure 1. Locally cusp edges belong to the closure of two sheets of L that we refer to as the upper and lower sheets of the cusp edge.
- Any point in L_2 has a neighborhood whose front projection is diffeomorphic to a standard **swallow tail** singularity. In fact, there are two types of swallow tail singularities that we refer to as **upward** and **downward** swallow tails. The Legendrian, L_F , in $J^1\mathbb{R}^2$ defined by the generating family,

$$F : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}, \quad F(x_1, x_2; e) = e^4 - x_1 e^2 + x_2 e,$$

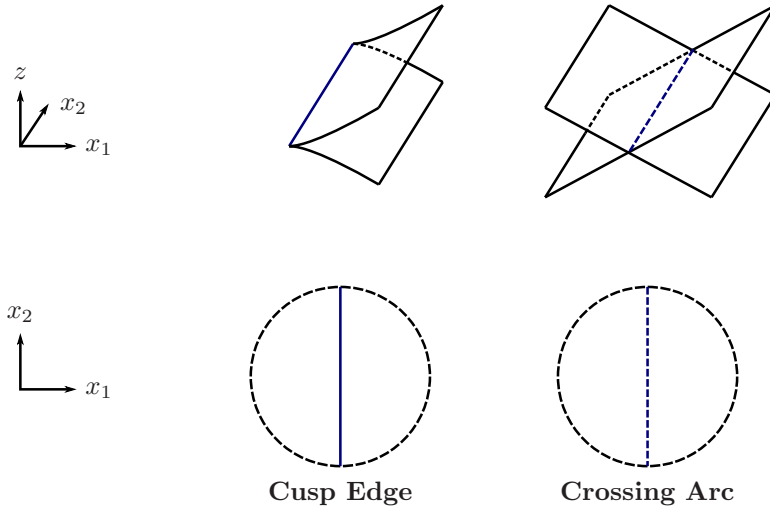


FIGURE 1. Codimension 1 parts of the singular set pictured in the front and base projection. In the base projection we picture crossing arcs (resp. cusp edges) as dotted (resp. solid) lines.

has a standard upward swallow tail singularity when $x_1 = x_2 = 0$. The front projection of L_F is given by

$$\pi_{xz}(L_F) = \{(x_1, x_2, f_{x_1, x_2}(e)) \mid (f_{x_1, x_2})'(e) = 0\} \quad \text{where } f_{x_1, x_2}(e) = F(x_1, x_2; e).$$

The standard downward swallow tail singularity is obtained by negating the z -coordinate. See [1, p. 47].

A swallow tail point, s_0 , lies in the closure of two cusp edges of $\pi_{xz}(L)$; the base projections of these two cusp edges form a semi-cubical cusp curve in S with the cusp point at $\pi_x(s_0)$. The two cusp edges that meet at a swallowtail point have either their upper or lower sheet in common. (The other two sheets associated with these cusp edges meet at the crossing arc that terminates at s_0 .) When the swallowtail is *upward* (resp. *downward*) the common sheet appears above (resp. below) the two sheets that cross in the front projection.

As long as $L \subset J^1S$ is an *embedded* Legendrian, the front projection of L_0 must be transverse to itself. [Since the y_0 and y_1 coordinates are recovered by $y_i = \partial z / \partial x_i$.] Again, after a small perturbation, we can assume L_0 is also transverse to L_1 and L_2 as well as to the self intersection set of L_0 to arrange that the self intersections of $\pi_{xz}(L)$ are as follows:

- Along **crossing arcs** two sheets of L intersect transversally.
- At isolated **triple points** three sheets of L meet in a manner that is pairwise transverse, and so that the crossing arc between any two of the sheets is transverse to the third sheet. In the base projection, three crossing arcs meet at a single point, but are pairwise transverse.
- At isolated points, x , a **cusp-sheet intersection** occurs where a cusp edge cuts transversally through a sheet of L . The crossing arcs between this sheet and the upper and lower sheets of the cusp edge both end at x . The base projections of these two crossing arcs meet at a semi-cubical cusp point on the base projection of the cusp edge.

We refer to the union of the crossing arcs, cusp edges, and swallowtail points in either the front or base projection (depending on context) as the **singular set** of L . We denote the base projection of the singular set as Σ . When L has generic base projection, we write

$$\Sigma = \Sigma_1 \sqcup \Sigma_2$$

where points in Σ_1 are in the image of a single crossing arc or cusp edge, and $\Sigma_2 \subset S$ is a finite set containing the image of codimension 2 parts of the front projection as well as points in the transverse intersection of the image of two distinct cusp edges and/or crossing arcs. Often, we will refer to the base projections of cusp edges, and crossing arcs as the **cusp locus** and the **crossing locus**. See Figures 1 and 2 for the local appearance of the singular set.

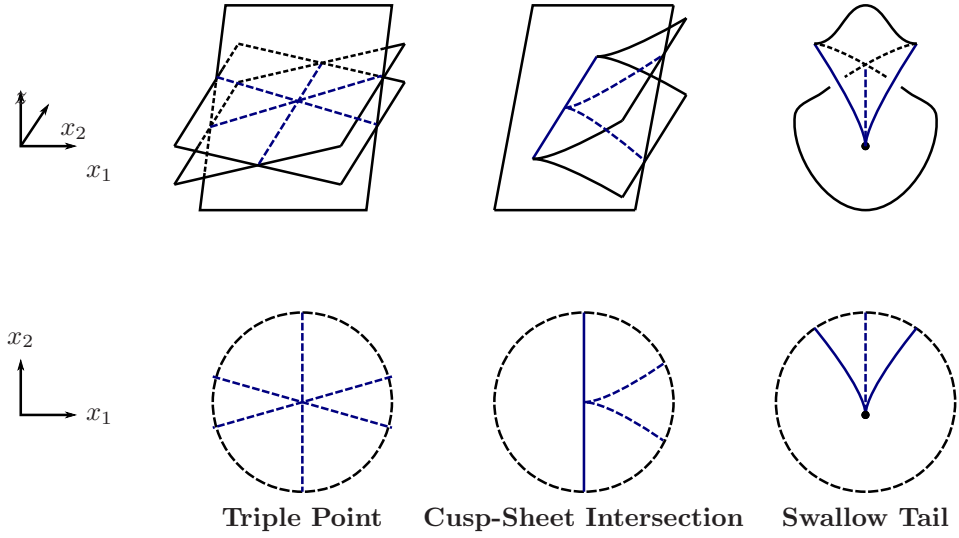


FIGURE 2. Codimension 2 parts of the singular set pictured in the front and base projection. Note that the pictured swallow tail is *upward*. The front projection of a downward swallow tail is obtained by reflecting the z -direction. Additional codimension 2 points exist in the base projection where two crossing and/or cusp arcs intersect transversally.

Conversely, any surface in $S \times \mathbb{R}$ matching the above description and without vertical tangent planes is the front projection of a unique Legendrian $L \subset J^1 S$. See Figure 3 for an example of a Legendrian surface in $J^1(\mathbb{R}^2)$ pictured in its front and base projection.

2.2.1. Maslov potentials. A generic loop γ in L is disjoint from swallowtail points and crosses cusp edges transversally. Let $m(\gamma) = D(\gamma) - U(\gamma)$ where $D(\gamma)$ (resp. $U(\gamma)$) is the number of points where γ crosses from the upper to lower sheet (resp. lower to upper sheet) at a cusp edge. The (minimal) **Maslov number** for L , $m(L) \in \mathbb{Z}_{\geq 0}$ is the smallest positive integer value taken by $m(\gamma)$, or 0 if all loops have $m(\gamma) = 0$.

A **Maslov potential** is a locally constant function

$$\mu : L_0 \rightarrow \mathbb{Z}/m(L)$$

whose value increases by 1 when passing from the lower sheet to the upper sheet at a cusp edge. When L is connected, any two Maslov potentials differ by the overall addition of a constant.

3. DEFINITION OF THE CELLULAR DGA (\mathcal{A}, ∂)

In this section we associate a DGA to a Legendrian surface, $L \subset J^1 S$, using an appropriately chosen cell decomposition of S . Initially we assume that the front projection of L does not contain any swallowtail singularities as this simplifies the definition. In Sections 3.7-3.13, we complete the definition by addressing the case when swallowtail points are present.

3.1. Compatible cell decompositions. Let S be a surface. By a **polygonal decomposition** of S , we mean a CW-complex decomposition of S ,

$$S = \sqcup_{i=0}^2 \sqcup_{\alpha} e_{\alpha}^i$$

with characteristic maps $c_{\alpha}^i : D^i \rightarrow S$ satisfying:

- (i) The characteristic maps for 1-cells are smooth.
- (ii) For any two cell e_{α}^2 , preimages of 0-cells divide the boundary of D^2 into intervals that are mapped homeomorphically to 1-cells by c_{α}^2 .

Note that we allow that the same 1-cell may appear multiple times in the boundary of a given 2-cell. Compare with the decomposition pictured in Figure 11.

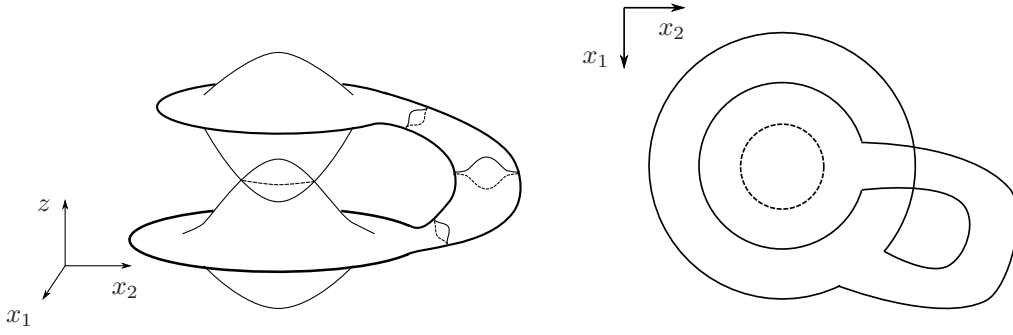


FIGURE 3. Example of a Legendrian surface in $J^1(\mathbb{R}^2)$ and its base projection.

Let $L \subset J^1 S$ be a Legendrian submanifold with generic front projection, and denote by $\Sigma = \Sigma_1 \sqcup \Sigma_2 \subset S$ the base projection of the singular set of L , where Σ_1 and Σ_2 are the singularities of codimension 1 and 2 respectively. We say that a polygonal decomposition $\mathcal{E} = \{e_\alpha^i\}$ of S is **compatible** with L or L -compatible if Σ is contained in the 1-skeleton of \mathcal{E} . From the nature of codimension 2 singularities, it follows that Σ_2 must be contained in the 0-skeleton.

3.2. Definition of the cellular DGA (\mathcal{A}, ∂) . Assume that \mathcal{E} is a compatible polygonal decomposition for a Legendrian L whose front projection is without swallowtail points. We now associate a differential graded algebra, (\mathcal{A}, ∂) , to L using \mathcal{E} . We will refer to (\mathcal{A}, ∂) as the **cellular DGA** of L , and in Section 3.7 we extend the definition to allow swallowtail points.

The definition of (\mathcal{A}, ∂) requires the following additional data associated to \mathcal{E} . For each 2-cell, e_α^2 , we choose an **initial and terminal vertex** from points of S^1 that are mapped to 0-cells by $c_\alpha^2|_{S^1}$, and label these points v_0^α and v_1^α . We allow that $v_0^\alpha = v_1^\alpha$, but in this case we must also declare a direction for the path around the circle from v_0^α to v_1^α .

3.3. The algebra \mathcal{A} . When referring to **sheets** of L above a cell $e_\alpha^i \in \mathcal{E}$ we mean those components of $L \cap \pi_x^{-1}(e_\alpha^i)$ that are not contained in a cusp edge. We denote the set of sheets of L above e_α^i as $L(e_\alpha^i) = \{S_p \mid p \in I_\alpha\}$ where I_α is an indexing set. The sets $L(e_\alpha^i)$ are partially ordered by decreasing z -coordinate, and we write $S_p \prec S_q$ if $z(S_p) > z(S_q)$ above e_α^i . Two sheets are incomparable if and only if they meet in a crossing arc above e_α^i in the front projection of L .

The algebra \mathcal{A} is the free unital associative (non-commutative) $\mathbb{Z}/2$ -algebra with generating set arising from the cells of \mathcal{E} as follows. For each cell e_α^i we associate one generator for each pair of sheets $S_p, S_q \in L(e_\alpha^i)$ satisfying $S_p \prec S_q$. We denote these generators as $a_{p,q}^\alpha$, $b_{p,q}^\alpha$, or $c_{p,q}^\alpha$ in the case of a 0-cell, 1-cell, or 2-cell respectively. The superscript α will sometimes be omitted from notation.

3.4. The grading on \mathcal{A} . A choice of Maslov potential, μ , for L allows us to assign a $\mathbb{Z}/m(L)$ -grading to \mathcal{A} as follows (where $m(L)$ is the Maslov number of L). Each of the generators is a homogeneous element with degree given by

$$(1) \quad |c_{p,q}^\alpha| = \mu(S_p) - \mu(S_q) + 1; \quad |b_{p,q}^\alpha| = \mu(S_p) - \mu(S_q); \quad \text{and} \quad |a_{p,q}^\alpha| = \mu(S_p) - \mu(S_q) - 1.$$

When L is connected the grading is independent of the choice of μ .

3.5. Algebra in $Mat(n, \mathcal{A})$. In the remainder of the article, we often make computations in the ring, $Mat(n, \mathcal{A})$, of $n \times n$ matrices with entries in \mathcal{A} . Any linear map $f : \mathcal{A} \rightarrow \mathcal{A}$ extends to a linear map $f : Mat(n, \mathcal{A}) \rightarrow Mat(n, \mathcal{A})$ by applying f entry-by-entry. Moreover, if f is an algebra homomorphism (resp. a derivation), then the resulting extension is also an algebra homomorphism (resp. a derivation). Note that a derivation of \mathcal{A} with $\partial(1) = 0$ will annihilate any matrix of constants in $Mat(n, \mathbb{Z}/2) \subset Mat(n, \mathcal{A})$, so that for instance we can compute $\partial(PBP^{-1}) = P(\partial B)P^{-1}$ if $P \in GL(n, \mathbb{Z}/2)$.

In our notation, we will use $E_{i,j}$ to denote a matrix with all entries 0 except for the (i, j) -entry which is 1.

3.6. Defining the differential. We define the differential, ∂ , by requiring $\partial(1) = 0$; specifying values on the generators of \mathcal{A} ; and then extending ∂ as a derivation. Generators that correspond to cells of dimension 0, 1 and 2 are considered separately in the definition.

3.6.1. *0-cells.* For a zero cell, e_α^0 , extend the partial ordering \prec to a total linear order via a bijection $\iota : \{1, \dots, n\} \rightarrow I_\alpha$, so that the (non-cusp) sheets above e_α^0 are labeled as $S_{\iota(1)}, \dots, S_{\iota(n)}$ with $z(S_{\iota(1)}) \geq \dots \geq z(S_{\iota(n)})$. Using this ordering, we arrange the generators corresponding to e_α^0 into a strictly upper triangular $n \times n$ matrix A with (i, j) -entry given by $a_{\iota(i), \iota(j)}^\alpha$ if $S_{\iota(i)} \prec S_{\iota(j)}$ and 0 otherwise. The differential is then defined so that the matrix equation

$$(2) \quad \partial A = A^2$$

holds with ∂ applied entry-by-entry.

Lemma 3.1. *There is a unique way to define $\partial a_{p,q}^\alpha$ so that (2) holds. Moreover, $\partial a_{p,q}^\alpha$ is independent of the choice of extension of \prec to a total order, and $\partial^2 a_{p,q}^\alpha = 0$.*

Proof. Uniqueness is clear since ∂A contains each $\partial a_{p,q}^\alpha$ as an entry. That such a definition is possible requires that all non-zero entries of A^2 correspond to non-zero entries of A . Using $A = (x_{i,j})$ for the entries of A , the (i, j) -entry of A^2 is $\sum_k x_{i,k} x_{k,j} = \sum_k a_{\iota(i), \iota(k)}^\alpha a_{\iota(k), \iota(j)}^\alpha$ where the latter sum is over those k such that $S_{\iota(i)} \prec S_{\iota(k)} \prec S_{\iota(j)}$ and hence vanishes unless $S_{\iota(i)} \prec S_{\iota(j)}$. It is also clear that the sum depends only on \prec and not the choice of extension to a total order.

To see that $\partial^2 a_{p,q}^\alpha = 0$, compute

$$\partial^2 A = \partial(A^2) = (\partial A)A + A(\partial A) = A^3 + A^3 = 0.$$

□

3.6.2. *1-cells.* For generators corresponding to a 1-cell, e_α^1 , extend \prec to a total order via a bijection $\iota : \{1, \dots, n\} \rightarrow I_\alpha$ and form an $n \times n$ matrix $B = (x_{i,j})$ with $x_{i,j} = \begin{cases} b_{\iota(i), \iota(j)}^\alpha & \text{if } S_{\iota(i)} \prec S_{\iota(j)}, \\ 0 & \text{else} \end{cases}$. The characteristic map for e_α^1 allows us to refer to the 0-cells at the boundary of e_α^1 as initial and terminal vertices, and we denote these cells as e_- and e_+ . (It may be the case that $e_- = e_+$.) The values, $\partial b_{\iota(i), \iota(j)}^\alpha$, are determined by the matrix equation

$$(3) \quad \partial B = A_+(I + B) + (I + B)A_-$$

where A_\pm are $n \times n$ matrices formed from the generators corresponding to e_\pm in a manner that will be described presently.

For convenience of notation, we only define A_+ as the definition of A_- is identical. Each sheet above e_+ belongs to the closure of a unique sheet of $L(e_\alpha^1)$. Since we have already chosen a bijection, $\{1, \dots, n\} \cong L(e_\alpha^1)$, this produces an order preserving injection $\kappa : L(e_+) \hookrightarrow \{1, \dots, n\}$. Those sheets of e_α^1 not in the image of κ meet in pairs at cusp points above e_+ . We form A_+ so that the (i, j) entry is $a_{p,q}^+$ if $S_p \prec S_q$ and $\kappa(p) = i$, $\kappa(q) = j$; the $(k, k+1)$ entry is 1 if the k and $k+1$ sheets of $L(e_\alpha^1)$ meet at a cusp above e_+ ; and all other entries are 0. Alternatively, one can think of using the total ordering arising from κ to form a matrix out of the $a_{p,q}^+$ and then inserting 2×2 blocks of the form $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

along the diagonal for each pair of sheets of $L(e_\alpha^1)$ that meet at a cusp above e_+ . For example, if there

are 4 sheets above e_α^1 and the top two meet at a cusp at e_+ , then $A_+ = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{12}^+ \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Lemma 3.2. *There is a unique way to define $\partial b_{p,q}^\alpha$ so that (3) hold. Moreover, $\partial b_{p,q}^\alpha$ is independent of the extension of \prec to a total order, and $\partial^2 b_{p,q}^\alpha = 0$.*

Proof. In order to be able to define $\partial b_{p,q}^\alpha$ so that (3) will hold, we need to know that for any 0 entries of B , the corresponding entry of the right hand side of (3) is 0. This is only an issue when e_α^1 is in the base projection of a crossing arc as otherwise B is a full strictly upper triangular matrix, and in general the matrices A_\pm (resp. $(I + B)$) are strictly upper triangular (resp. upper triangular). Assuming the k and $k+1$ sheets cross above e_α^1 , it suffices to check that A_\pm is strictly upper triangular with the $(k, k+1)$ -entry equal to 0. This is clear since the sheets of $L(e_\pm)$ that correspond to the k and $k+1$ sheets of $L(e_\alpha^1)$ must also have the same z -coordinate above e_+ .

To check independence of $\partial b_{p,q}^\alpha$ from the choice of total order, we again only need to consider the case of a crossing arc above e_α^1 . The two choices of total order lead to B matrices B' and B'' that are related by conjugation by the permutation matrix Q associated to the transposition of the two sheets that cross. The corresponding A_\pm matrices A'_\pm and A''_\pm are related in the same manner. Therefore, the equations are equivalent since

$$\begin{aligned} \partial B' &= A'_+(I + B') + (I + B')A'_- \Leftrightarrow Q(\partial B')Q = Q[A'_+(I + B') + (I + B')A'_-]Q \Leftrightarrow \\ &\partial B'' = A''_+(I + B'') + (I + B'')A''_- \end{aligned}$$

where we use the observation from Section 3.5 in the 2-nd equivalence.

To verify $\partial^2 B = 0$, observe that in all cases $\partial A_\pm = A_\pm^2$ because of (2) combined with the block nature of the matrix A_\pm and the computation $\partial \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2$. We compute

$$\begin{aligned} \partial^2(I + B) &= \partial[A_+[I + B] + [I + B]A_-] = (\partial A_+)[I + B] + A_+[\partial(I + B)] + [\partial(I + B)]A_- + [I + B](\partial A_-) = \\ &(A_+^2)[I + B] + A_+[A_+(I + B) + (I + B)A_-] + [A_+(I + B) + (I + B)A_-]A_- + [I + B](A_-)^2 = 0. \end{aligned}$$

□

3.6.3. 2-cells. For a 2-cell, e_α^2 , the partial ordering of sheets in $L(e_\alpha^2)$ is a total ordering, so we take $\{1, \dots, n\}$ for the indexing set I_α and label sheets as S_1, \dots, S_n with $z(S_1) > \dots > z(S_n)$. Using this ordering of sheets above e_α^2 , we identify the sheets above all of the edges and vertices that appear along the boundary of e_α^2 with subsets of $\{1, \dots, n\}$. Then, for each such edge (resp. vertex) we can place the corresponding generators $b_{p,q}^\alpha$ (resp. $a_{p,q}^\alpha$) into an $n \times n$ matrices with 2×2 blocks of the form $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (resp. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$) inserted in the $k, k+1$ position along the diagonal whenever sheets $S_k, S_{k+1} \in L(e_\alpha^2)$ meet at a cusp above the edge (resp. vertex).

Recall that for each 2-cell we have chosen initial and terminal vertices, v_0^α and v_1^α , along the boundary of D^2 that are mapped to 0-cells of \mathcal{E} via the characteristic map, c_α^2 , for the 2-cell. Let γ_+ and γ_- denote paths in S^1 that proceed counter-clockwise and clockwise respectively from v_0^α to v_1^α . (If $v_0^\alpha = v_1^\alpha$, then we choose one of these paths to be constant and the other to be the entire circle as mentioned in Section 3.2.) We let B_1, \dots, B_j (resp. B_{j+1}, \dots, B_m) denote the matrices associated (as in the previous paragraph) to the successive edges of \mathcal{E} that appear in the image of $c_\alpha^2 \circ \gamma_+$ (resp. $c_\alpha^2 \circ \gamma_-$). In addition, we let A_{v_0} and A_{v_1} be the matrices associated (as in the previous paragraph) to the initial and terminal vertices v_0^α and v_1^α . Collecting the generators corresponding to e_α^2 into the strictly upper triangular matrix C , we define $\partial c_{i,j}^\alpha$ so that

$$(4) \quad \partial C = A_{v_1}C + CA_{v_0} + (I + B_j)^{\eta_j} \dots (I + B_1)^{\eta_1} + (I + B_m)^{\eta_m} \dots (I + B_{j+1})^{\eta_{j+1}}$$

where the exponent, η_i , is 1 (resp. -1) if the parametrization of the edge corresponding to B_i given by $c_\alpha^2 \circ \gamma_\pm$ has the same (resp. opposite) orientation as the characteristic map of the 1-cell. See Figure 4.

Lemma 3.3. *We can uniquely define $\partial c_{i,j}^\alpha$ so that (4) holds. Moreover, $\partial^2 C = 0$.*

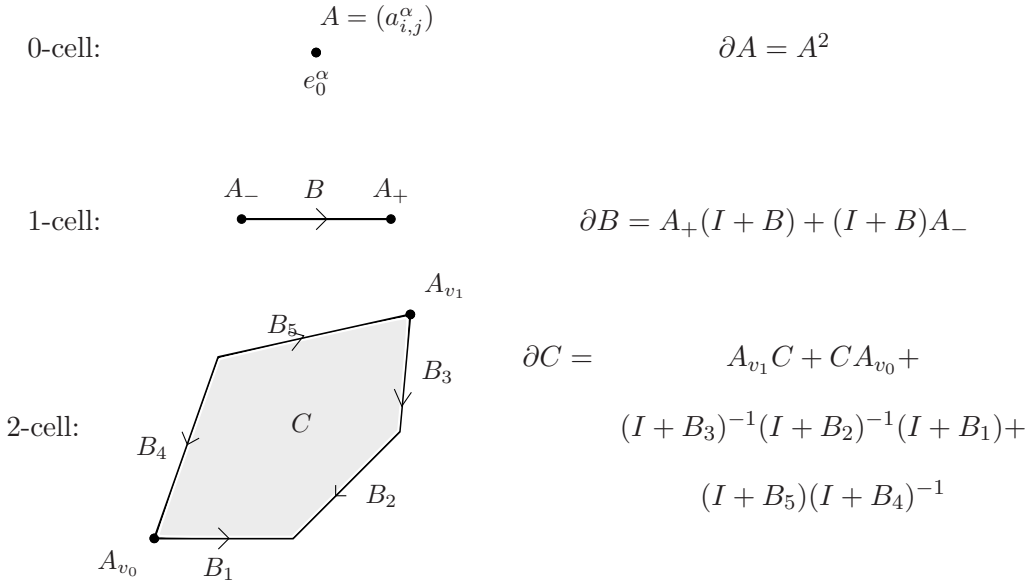
Proof. First, observe that since the matrices B_i are strictly uppertriangular, and hence nilpotent in $\text{Mat}(n, \mathcal{A})$, the inverse of $(I + B_i)$ is given by the finite geometric series

$$(I + B_i)^{-1} = I + B_i + B_i^2 + \dots$$

That the equation (4) may be obtained relies on the right hand side being strictly upper triangular. This follows since the first two terms are strictly upper triangular and the 3-rd and 4-th terms both have the form $I + X$ where X is strictly upper triangular.

Next, each B_i corresponds to a 1-cell, e_i^1 , in \mathcal{E} . Moreover, e_i^1 has initial and terminal vertices (using the orientation of e_i^1 to distinguish initial from terminal). Denote by A_i^+ and A_i^- the $n \times n$ matrices associated to these vertices using the total ordering and insertion of 2×2 blocks specified by the 2-cell e_α^2 . We claim that

$$(5) \quad \partial B_i = A_i^+(I + B_i) + (I + B_i)A_i^-.$$

FIGURE 4. Summary of differentials in (\mathcal{A}, ∂) .

Indeed, as long as there are not two sheets of $L(e_\alpha^2)$ that meet at a cusp above e_i^1 , this is just (3) with matrices formed using the total order from $L(e_\alpha^2)$. If it does happen that $S_k, S_{k+1} \in L(e_\alpha^2)$ share a cusp edge in their closure above e_i^1 , then the matrix B_i , (resp. A_i^\pm) is obtained from the corresponding matrix in (3) by forming a block matrix with $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (resp. $N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$) inserted at the $k, k+1$ part of the diagonal of B_i (resp. A_i^\pm). Then, using block matrix calculations, (5) follows from (3) and the calculation for 2×2 matrices

$$\partial 0 = N(I + 0) + (I + 0)N.$$

From (5), we can deduce that

$$(6) \quad \partial[(I + B_i)^{-1}] = A_i^-(I + B_i)^{-1} + (I + B_i)^{-1}A_i^+$$

by applying the Liebniz rule to $0 = \partial(I) = \partial[(I + B)(I + B)^{-1}]$ and solving for $\partial[(I + B_i)^{-1}]$.

This gives us that $\partial(I + B_i)^{\eta_i} = X_i^+(I + B_i)^{\eta_i} + (I + B_i)^{\eta_i}X_i^+$ where the X_i^\pm correspond to the initial and terminal vertex of e_i^1 with respect to the orientation given by γ_+ or γ_- rather than by the orientation of e_i^1 itself. That is, $X_i^\pm = A_i^\pm$ if $\eta_i = 1$, and $X_i^\pm = A_i^\mp$ if $\eta_i = -1$. Since $X_i^+ = X_{i+1}^-$ for $1 \leq i \leq j-1$ and $j+1 \leq i \leq m$, the sums arising from expanding $\partial[(I + B_j)^{\eta_j} \cdots (I + B_1)^{\eta_1}]$ and $\partial[(I + B_m)^{\eta_m} \cdots (I + B_{j+1})^{\eta_{j+1}}]$ using the Liebniz rule telescope to give

$$\begin{aligned} \partial[(I + B_j)^{\eta_j} \cdots (I + B_1)^{\eta_1}] &= X_j^+(I + B_j)^{\eta_j} \cdots (I + B_1)^{\eta_1} + (I + B_j)^{\eta_j} \cdots (I + B_1)^{\eta_1} X_1^- = \\ &A_{v_1}(I + B_j)^{\eta_j} \cdots (I + B_1)^{\eta_1} + (I + B_j)^{\eta_j} \cdots (I + B_1)^{\eta_1} A_{v_0}, \end{aligned}$$

and

$$\partial[(I + B_m)^{\eta_m} \cdots (I + B_{j+1})^{\eta_{j+1}}] = A_{v_1}(I + B_m)^{\eta_m} \cdots (I + B_{j+1})^{\eta_{j+1}} + (I + B_m)^{\eta_m} \cdots (I + B_{j+1})^{\eta_{j+1}} A_{v_0}.$$

With these formulas in hand, $\partial^2 C = 0$ follows in a straightforward manner from (4). \square

Explicit examples computing this differential appear in Section 5.

3.7. Extending the definition to allow swallowtail points. For a Legendrian L with swallowtail points and compatible polygonal decomposition \mathcal{E} , the DGA (\mathcal{A}, ∂) is defined in the same general manner. For a cell whose closure is disjoint from the base projections of swallowtail points, generators and differentials are defined as in Section 3.2. The appropriate modifications of the definition for 0-cells, 1-cells, and 2-cells that border swallow tail points are given below in 3.10, 3.12, and 3.13.

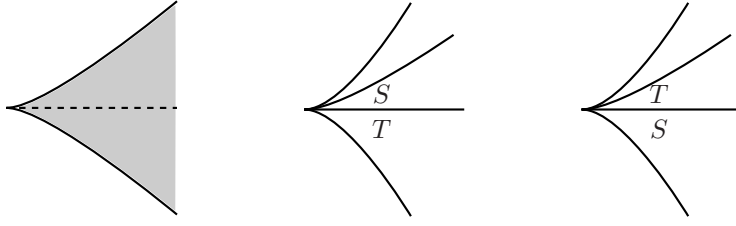


FIGURE 5. (left) The base projection of the singular set at a swallowtail point. Solid curves denote cusp arcs, and the dotted curve is a crossing arc. The swallowtail region is shaded. (right) An example of an L -compatible cellular decomposition near a swallowtail point with the two possible labelings by S and T . Here one of the 1-cells is not on the singular set.

3.8. Local appearance near a swallowtail point. Suppose that e_α^0 is the base projection of a swallowtail point of L . The singular locus $\Sigma \subset S$ near e_α^0 consists of 3 arcs that share a common end point at e_α^0 . Two of these arcs are projections of cusp edges that meet in a semi-cubical cusp point at e_α^0 , and the third is a crossing arc that lies between the cusp arcs. See Figure 5. In a neighborhood of e_α^0 , we refer to the region between the two cusp arcs (right of the cusp point in Figure 5) as the **swallowtail region**. Note that there are 2 more sheets above the swallowtail region than there are above its complement.

Recall that the appearance of the front projection of L near a swallowtail point s_0 has one of two distinct types depending on if the s_0 is an *upward* or *downward* swallow tail. We use the convention of indexing the 3 sheets that meet at an upward (resp. downward) swallowtail point as $k, k+1, k+2$ (resp. $l-2, l-1, l$), so that k (resp. l) denotes the upper (resp. lower) sheet and the $k+1$ and $k+2$ (resp. $l-2$ and $l-1$) sheets cross near s_0 . The single sheet outside the swallowtail region that contains s_0 in its closure is then labeled k (resp. $l-2$).

3.9. Decorations at swallowtail points. Defining (\mathcal{A}, ∂) in the presence of swallowtail points requires some additional data. Within a neighborhood of each swallowtail point, $s \in \Sigma_2$, we choose a labeling of the two regions (subsets of 2-cells of \mathcal{E}) that border the crossing arc that ends at s ; label one of the regions with an S and the other with a T . See Figure 5.

3.10. 0-cells. In case the 0-cell, e_α^0 , is the projection of a swallowtail point the definition of the algebra does not require serious modification. Just note that we do consider the swallowtail point itself to be a sheet of $L(e_\alpha^0)$ so that there are the same number of generators $a_{i,j}^\alpha$ as if e_α^0 were located in the complement of the swallowtail region. The differential is defined by (2) as usual.

3.11. Matrices associated to a swallowtail point. For each swallow tail point, $s \in \Sigma_2$, we will use several matrices formed from the generators, $a_{i,j}$, associated to s (which is necessarily a 0-cell of \mathcal{E}). Near s , we suppose that there are n sheets above the swallow tail region, and $n-2$ in the complement of the swallow tail region. This is partly inspired by the swallowtail discussion of Cerf theory in [27].

The $n-2$ sheets above s itself are totally ordered, and we let A denote the $(n-2) \times (n-2)$ matrix with non-zero entries $a_{i,j}$. For $1 \leq m_1 < m_2 \leq n$, we let \hat{A}_{m_1, m_2} denote the $n \times n$ matrix obtained from A by inserting the block $N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ along the main diagonal at the (possibly non-consecutive) m_1 and m_2 rows and columns with the rest of the entries in these rows and columns equal to 0.

In the case of an *upward* swallowtail point that involves sheets $k, k+1$, and $k+2$, we use the following matrix notations:

$$\begin{aligned}
 A_S &= [I + E_{k+2, k+1}] \hat{A}_{k, k+2} [I + E_{k+2, k+1}]; & A_T &= [I + E_{k+1, k+2}] \hat{A}_{k, k+1} [I + E_{k+1, k+2}]; \\
 (7) \quad S &= I + \hat{A}_{k, k+1} E_{k+2, k} + E_{k+1, k+2} & T &= I + E_{k+1, k+2}. \\
 &= I + \sum_{i < k} a_{i, k} E_{i, k} + E_{k+1, k+2};
 \end{aligned}$$

In the case of a *downward* swallowtail point that involves sheets $l-2, l-1$, and l , we use

$$(8) \quad \begin{aligned} A_S &= [I + E_{l-1, l-2}] \hat{A}_{l-2, l} [I + E_{l-1, l-2}]; \quad A_T = [I + E_{l-2, l-1}] \hat{A}_{l-1, l} [I + E_{l-2, l-1}]; \\ S &= I + E_{l, l-2} \hat{A}_{l-1, l} + E_{l-2, l-1} \quad T = I + E_{l-2, l-1}. \\ &= I + \sum_{l-2 < j} a_{l-2, j} E_{l, j+2} + E_{l-2, l-1}; \end{aligned}$$

For instance, if there are 4-sheets above the swallow tail region of an upward swallow tail, s , that involves sheets 2, 3, and 4, then

$$\begin{aligned} A &= \begin{bmatrix} 0 & a_{12} \\ 0 & 0 \end{bmatrix}; \quad \hat{A}_{k, k+1} = \begin{bmatrix} 0 & 0 & 0 & a_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad \hat{A}_{k, k+2} = \begin{bmatrix} 0 & 0 & a_{12} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \\ A_S &= \begin{bmatrix} 0 & 0 & a_{12} & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad A_T = \begin{bmatrix} 0 & 0 & 0 & a_{12} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \\ S &= \begin{bmatrix} 1 & a_{12} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Remark 3.1. These somewhat mysterious matrices may be interpreted in a nice way using generating families. When L is defined by a generating family, $F : S \times \mathbb{R}^N \rightarrow \mathbb{R}$, the sheets of L above $x \in S$ correspond to the critical points of $F(x, \cdot) : \mathbb{R}^N \rightarrow \mathbb{R}$. Suppose that the entries, $a_{i, j}$, of A are replaced with the mod 2 count of gradient trajectories connecting sheet S_i to S_j near the swallow tail point, but outside of the swallowtail region. Then, the matrices S and T are continuation maps determined by the collection of handleslides that end at the swallowpoint as specified in [27]. The matrices A_S and A_T represent the differential in the Morse complex of F_x above a point x belonging to the crossing arc that ends at the swallowtail point, with respect to the ordering of basis vectors (i.e. sheets of L) as they appear above the regions decorated with S and T . See [36] for more details.

Lemma 3.4. *We have*

$$\partial A_S = (A_S)^2, \quad \partial A_T = (A_T)^2.$$

Moreover, for an upward swallowtail

$$(9) \quad \partial S = \hat{A}_{k, k+1} S + S A_S = S \hat{A}_{k, k+1} + A_S S, \quad \text{and} \quad \partial T = \hat{A}_{k, k+1} T + T A_T;$$

for a downward swallowtail

$$(10) \quad \partial S = \hat{A}_{l-1, l} S + S A_S = S \hat{A}_{l-1, l} + A_S S, \quad \text{and} \quad \partial T = \hat{A}_{l-1, l} T + T A_T.$$

Proof. We verify these formulas in the case of an upward swallowtail. The case of a downward swallowtail is similar.

Compute

$$\begin{aligned} \partial A_S &= [I + E_{k+2, k+1}] \partial \hat{A}_{k, k+2} [I + E_{k+2, k+1}] = [I + E_{k+2, k+1}] (\hat{A}_{k, k+2})^2 [I + E_{k+2, k+1}] = \\ &= [I + E_{k+2, k+1}] (\hat{A}_{k, k+2}) [I + E_{k+2, k+1}] [I + E_{k+2, k+1}] (\hat{A}_{k, k+2}) [I + E_{k+2, k+1}] = A_S^2 \end{aligned}$$

where the second equality follows since $\partial A = A^2$ and $\partial N = N^2$. That $\partial A_T = (A_T)^2$ is similar.

For the second equation of (9), we compute from definitions

$$\hat{A}_{k, k+1} T + T A_T = \hat{A}_{k, k+1} T + T (T \hat{A}_{k, k+1} T) = 0 = \partial T.$$

To establish the first equation of (9), begin by observing that, for Q the permutation matrix of the transposition $(k+1, k+2)$, we have

$$(11) \quad S = [Q + E_{k+2, k+2}] F \text{ where } F = I + E_{k+2, k+1} + \hat{A}_{k, k+1} E_{k+2, k}.$$

Indeed, $[Q + E_{k+2,k+2}]\hat{A}_{k,k+1}E_{k+2,k} = \hat{A}_{k,k+1}E_{k+2,k}$ because all entries below the $k-1$ row of $AE_{k+2,k}$ are 0, and $[Q + E_{k+2,k+2}][I + E_{k+2,k+1}] = I + E_{k+1,k+2}$. Next, note that

(12)

$$F\hat{A}_{k,k+1} = [I + E_{k+2,k+1} + \hat{A}_{k,k+1}E_{k+2,k}]\hat{A}_{k,k+1} = \hat{A}_{k,k+1} + 0 + \hat{A}_{k,k+1}E_{k+2,k+1} = \hat{A}_{k,k+1}F + \hat{A}_{k,k+1}^2E_{k+2,k}$$

where the second equality is due to the N block in the $k, k+1$ rows of $\hat{A}_{k,k+1}$. Finally, using (11) and (12) compute

$$\begin{aligned} S\hat{A}_{k,k+1} &= [Q + E_{k+2,k+2}]F\hat{A}_{k,k+1} = [Q + E_{k+2,k+2}]\hat{A}_{k,k+1}F + [Q + E_{k+2,k+2}]\hat{A}_{k,k+1}^2E_{k+2,k} = \\ &[I + E_{k+2,k+1}]Q\hat{A}_{k,k+1}(Q[I + E_{k+2,k+1}][I + E_{k+2,k+1}]Q)F + \partial S = \\ &[I + E_{k+2,k+1}]Q\hat{A}_{k,k+1}Q[I + E_{k+2,k+1}]S + \partial S = A_S S + \partial S. \end{aligned}$$

(In the 3-rd equality we used that Q and $I + E_{k+2,k+1}$ are both self inverse.)

This gives $\partial S = S\hat{A}_{k,k+1} + A_S S$. Since $S^2 = I$, multiplying on left and right by S gives $S(\partial S)S = \hat{A}_{k,k+1}S + SA_S$. One checks that $S(\partial S)S = \partial S$, so this completes the proof. \square

3.12. 1-cells. Suppose that a 1-cell e_α^1 has its initial vertex, e_-^0 , or terminal vertex, e_+^0 , at a swallowtail point. The generators, $b_{p,q}^\alpha$, associated to e_α^1 arise from the partially ordered set of sheets $L(e_\alpha^1)$ as usual, and the differential is again characterized by (3). However, some adjustment may be required in forming the matrices A_\pm . To simplify notation, we define A_- in case e_-^0 is a swallowtail point, s , and we note that an identical definition applies for A_+ if e_+^0 is a swallowtail point.

- Suppose e_α^1 is outside the interior of the swallowtail region near e_-^0 , including if e_α^1 lies in either of the cusp arcs ending at e_-^0 . We then form the matrices B and A_- as in 3.6.2. (No issues arise as the sheets of $L(e_\alpha^1)$ are then totally ordered by \prec , and the swallow tail point in $L(e_-^0)$ is in the closure of a unique sheet of $L(e_\alpha^1)$.)
- Suppose e_α^1 is in the interior of the swallowtail region, and e_α^1 is not contained in the crossing arc that terminates at e_-^0 . Then, $L(e_\alpha^1)$ is totally ordered, and we form B as in 3.6.2. In addition, we set

$$A_- = \hat{A}_{k,k+1} \text{ (resp. } A_- = \hat{A}_{l-1,l} \text{)}$$

if e_-^0 is an upward (resp. downward) swallow tail point. (That is, in the upward (resp. downward) case we form A_- as if the upper (resp. lower) 2 of the 3 swallowtail sheets above e_α^1 meet at a cusp above e_-^0 .)

- Suppose e_α^1 is contained in the crossing arc that ends at the swallowtail point above e_-^0 . In forming the matrix, B , there are two possible choices for extending the ordering of $L(e_\alpha^1)$ to a total order. If the total ordering used agrees with the ordering of sheets on the side of the swallow tail decorated with an S , then we set

$$(13) \quad A_- = A_S.$$

If instead the ordering agrees with the side of the swallow tail decorated with a T , then

$$(14) \quad A_- = A_T.$$

Lemma 3.5. *With B, A_- , and A_+ formed in this manner, equation (3) leads to a well defined definition of $\partial b_{p,q}^\alpha$, and $\partial^2 b_{p,q}^\alpha = 0$.*

Proof. First, we verify well-definedness. The total ordering of $L(e_\alpha^1)$ is uniquely determined except in the case where e_α^1 is contained in a crossing locus. Suppose that this is the case and that e_-^0 is a swallowtail point. (When e_+^0 is a swallowtail point the same argument applies.) Changing the choice of total ordering conjugates B by the permutation matrix, Q , of the transposition $(k+1 \ k+2)$. The corresponding definitions of A_- from (13) and (14) are related in the same manner since

$$\begin{aligned} QA_TQ &= Q[I + E_{k+1,k+2}]\hat{A}_{k,k+1}[I + E_{k+1,k+2}]Q = Q[I + E_{k+1,k+2}]Q(Q\hat{A}_{k,k+1}Q)Q[I + E_{k+1,k+2}]Q = \\ &[I + E_{k+2,k+1}]\hat{A}_{k,k+2}[I + E_{k+2,k+1}] = A_S. \end{aligned}$$

Thus, as in the proof of Lemma 3.2, the two versions of (3) that arise from different choices of ordering of $L(e_\alpha^1)$ are equivalent.

Since the $k+1$ and $k+2$ sheets cross above e_α^1 , it is also necessary to check that the $(k+1, k+2)$ entry of the right hand side of (3) is 0. Here, it is enough to check that A_- is strictly upper triangular with the $(k+1, k+2)$ -entry equal to 0. (This implies that $(I+B)A_-$ has the desired property, and a similar argument or the argument of Lemma 3.2 applies to the $A_+(I+B)$ term.) Since either ordering produces an equivalent equation, we can assume that the total ordering for sheets of $L(e_\alpha^1)$ coincides with the order of sheets in the T region. Then, we can compute the 3×3 diagonal block at rows $k, k+1, k+2$ of $A_- = [I + E_{k+1, k+2}] \hat{A}_{-, k, k+1} [I + E_{k+1, k+2}]$ to be

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The computation $\partial^2 B = 0$ goes through as before since, according to Lemma 3.4, it is still the case that $\partial A_\pm = A_\pm^2$. \square

3.13. 2-cells. For a 2-cell, e_α^2 , the boundary of the domain of the characteristic map $c_\alpha^2 : D^2 \rightarrow \overline{e_\alpha^2}$ can be viewed as a polygon, P , with edges mapping to 1-cells and vertices mapping to 0-cells of \mathcal{E} . Some vertices may have their sufficiently small neighborhoods in D^2 mapping to a region near a swallowtail point that is labeled with an S or T . We indicate this pictorially by placing an S or T near the corresponding corner of P , and we form a new polygon Q by adding an extra edge that cuts out each such corner. Thus, the edges of Q each correspond to either a 1-cell of \mathcal{E} or an S or T region at a swallowtail point. Those edges corresponding to 1-cells receive orientations according to the orientation of the 1-cells (provided by characteristic maps). We orient the new S and T edges so that the orientation points away from the endpoint shared with the crossing arc that terminates at the swallowtail point. See Figure 6.

Next, we assign a matrix A_v to each vertex v of Q . We identify each such vertex with a 0-cell in \mathcal{E} by using the characteristic map of e_α^2 and declaring that both endpoints of S and T edges are identified with the swallowtail point of the corresponding S or T region. As before, we use the total ordering of sheets above e_α^2 and the nature of the singular set above v to form a matrix A_v from the generators of the 0-cell v with the following modifications made when v is a swallowtail point.

- Suppose v is the initial vertex of an S or T edge with respect to the orientation of that edge. Then, we respectively set $A_v = A_S$ or $A_v = A_T$.
- Suppose v is the terminal vertex of an S or T edge or that small neighborhoods of v are mapped within the swallowtail region, but not to one of the regions S or T . Then, if the swallowtail is upward $A_v = \hat{A}_{k, k+1}$, and if the swallowtail is downward $A_v = \hat{A}_{l-1, l}$. (That is, we form A_v as if v were located on the portion of the cusp edge at the swallowtail point that is in the same half of the swallowtail region as the image of a neighborhood of v .)
- If the 2-cell is in the complement of the swallowtail region, then no adjustment is required to define A_v . (That is, $A_v = A$.)

Next, we assign a matrix Y_e to each edge of Q as follows.

- Suppose e is an edge corresponding to a 1-cell of \mathcal{E} . Then, we use the total ordering of the sheets above e_α^2 and the nature of the singular set above e to place the generators $b_{i,j}$ associated to e into an $n \times n$ matrix B_e . This is precisely as in Section 3.6.3. We then set

$$Y_e = I + B_e.$$

- Suppose e is an edge corresponding to an S region at a swallowtail point. Then, we take $Y_e = S$.
- Suppose e is an edge corresponding to a T region. Then, we take $Y_e = T$.

We form an upper triangular matrix C from the generators $c_{i,j}^\alpha$ associated to e_α^2 . To define ∂C , we make a choice of an initial and terminal vertex v_0 and v_1 from the vertices of Q . [Once again, if $v_0 = v_1$, then a direction needs to be chosen for the path around ∂Q from v_0 to v_1 .] We let γ_+ and γ_- denote paths around ∂Q that respectively proceed counter-clockwise and clockwise from v_0 to v_1 . [If $v_0 = v_1$, one of these paths is constant as specified by the choice of direction from v_0 to v_1 .]

Let Y_1, \dots, Y_j (resp. Y_{j+1}, \dots, Y_m) denote the matrices Y_e associated to successive edges of Q that appear along γ_+ (resp. γ_-), and let A_{v_0} and A_{v_1} be the matrices associated to the vertices v_0 and v_1 .

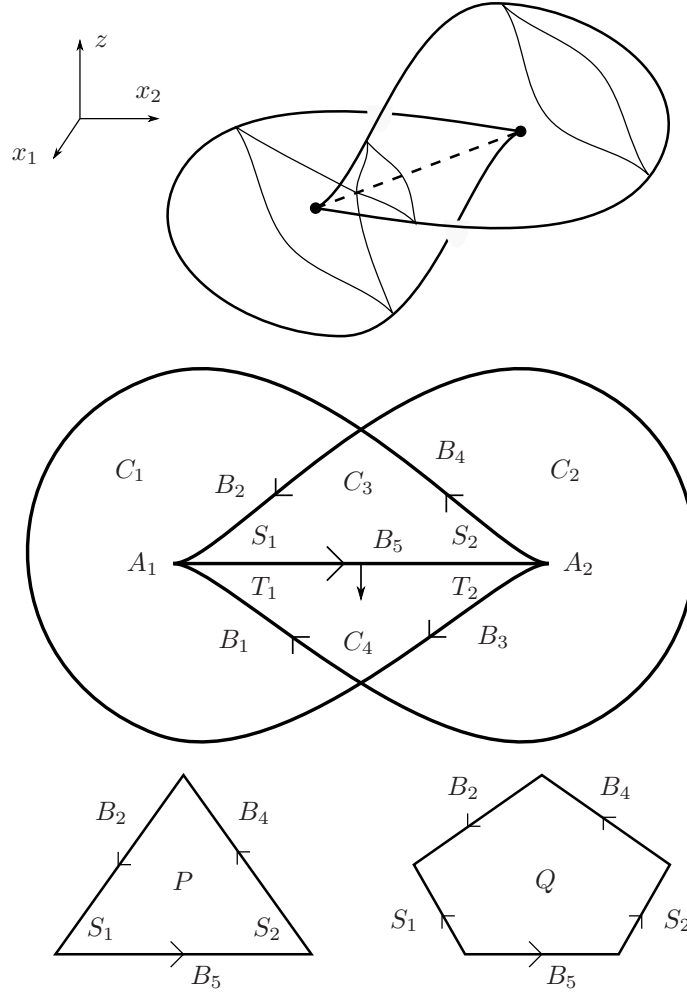


FIGURE 6. (top) The front projection of a Legendrian, L , with two swallowtail points. The swallowtail point on the left (resp. right) is upward (resp. downward). (middle) A compatible cell decomposition of the base projection of L with choices of S and T decorations made at swallowtail points. (bottom) The polygon Q used in defining ∂C_3 is formed by adding additional edges at all S and T corners of P .

We define $\partial c_{i,j}^\alpha$ so that

$$(15) \quad \partial C = A_{v_1} C + C A_{v_0} + Y_j^{\eta_j} \cdots Y_1^{\eta_1} + Y_m^{\eta_m} \cdots Y_{j+1}^{\eta_{j+1}}$$

where the exponent, η_i , is 1 (resp. -1) if the orientation of the corresponding edge of Q agrees (resp. disagrees) with the orientation of γ_\pm . Here, it is useful to note that $S = S^{-1}$ and $T = T^{-1}$.

Lemma 3.6. *Equation (4) leads to a well defined definition of $\partial c_{p,q}^\alpha$, and $\partial^2 c_{p,q}^\alpha = 0$.*

Proof. Equation (15) may be used to define $\partial c_{i,j}^\alpha$, since the matrices A_{v_i} and C are strictly upper triangular and the Y_i are upper triangular with 1's on the diagonal. [Thus, both products $Y_j^{\eta_j} \cdots Y_1^{\eta_1}$ and $Y_m^{\eta_m} \cdots Y_{j+1}^{\eta_{j+1}}$ have the form $I + X$ with X strictly upper triangular, so that the entries on the main diagonal cancel.]

Recall that matrices have been assigned to all edges and vertices of Q . For an edge, e , of Q let Y_e , A_{e^-} and A_{e^+} denote the matrices so assigned to e and the initial and terminal vertices e^- and e^+ of e (with respect to the orientation of e). We claim that

$$(16) \quad \partial Y_e = A_{e^+} Y_e + Y_e A_{e^-}.$$

Note that $\partial^2 C = 0$ can then be checked using an argument similar to the proof of Lemma 3.3 with (16) used in place of (5). In the case where e corresponds to a 1-cell of \mathcal{E} , (16) follows from observing the relation between the matrices Y_e and A_{e^\pm} , formed when viewing e and e^\pm as belonging to the boundary of Q , and the matrices $I + B$ and A_\pm from (3) that are used in defining the differential of

the generators associated to e . As in the proof of Lemma 3.3, if it is not the case that two sheets of e_α^2 meet at a cusp above e , then $Y_e = I + B$ and $A_{e^\pm} = A_\pm$ provided that, in forming $I + B$ and A_\pm , we use the ordering of sheets above e_α^2 and follow the provisions of 3.12.

Finally, if e is an edge of Q labeled with an S or T , then (16) is one of the equations in (9) (resp. in (10)) from Lemma 3.4 if the swallowtail is upward (resp. downward). \square

An explicit example computing this differential with swallowtails appears in Section 5.2.

We summarize the results of Section 3.

Theorem 3.2. *The cellular DGA (\mathcal{A}, ∂) satisfies $\partial^2 = 0$. A choice of Maslov potential, μ , on L provides a $\mathbb{Z}/m(L)$ -grading on \mathcal{A} for which ∂ has degree -1 .*

Proof. That $\partial^2 = 0$ has been verified during the definition of (\mathcal{A}, ∂) . Using equation (1) it is straightforward to verify that ∂ has degree -1 . \square

4. INDEPENDENCE OF CELL DECOMPOSITION

In this section we prove the following:

Theorem 4.1. *The stable tame isomorphism type of the cellular DGA (\mathcal{A}, ∂) is independent of the choice of cell decomposition \mathcal{E} and additional data.*

This requires showing independence of the following items:

- (1) The orientation of 1-cells.
- (2) The choice of initial and terminal vertex for each 2-cell.
- (3) The choice of decorations at swallow tail points.
- (4) The choice of cell decomposition \mathcal{E} .

These results are obtained in Corollary 4.1, Corollary 4.2, and Theorem 4.6 below. Before embarking upon their proof we collect some algebraic preliminaries.

4.1. Ordering of generators. Our main tool for producing stable tame isomorphisms will be Theorem 2.1 whose application requires a DGA to have its generating set ordered so that the differential becomes triangular. Such orderings are provided for the cellular DGA in the following Lemma 4.1.

The cellular DGA (\mathcal{A}, ∂) was constructed in Section 3 with a specific generating set. Define a partial ordering of this generating set, $\prec_{\mathcal{A}}$, by declaring that $y \preceq_{\mathcal{A}} x$ if

- (1) The cell corresponding to x has larger dimension than the cell corresponding to y , or
- (2) The same cell, e_α^i , corresponds to both x and y , and subscripts p_1, q_1 and p_2, q_2 for x and y are such that $S_{p_1} \preceq S_{p_2}$ and $S_{q_2} \preceq S_{q_1}$ holds in $L(e_\alpha^i)$.

Lemma 4.1. *The differential of (\mathcal{A}, ∂) is triangular with respect to any ordering of the generating set that extends $\prec_{\mathcal{A}}$.*

Proof. As in the proof of Lemma 3.1, we have $\partial a_{p,q}^\alpha = \sum a_{p_r}^\alpha a_{r_q}^\alpha$ with the sum over those sheets $S_r \in L(e_\alpha^0)$ such that $S_p \prec S_r \prec S_q$. Therefore, all of these generators satisfy $a_{p_r}^\alpha, a_{r_q}^\alpha \prec_{\mathcal{A}} a_{p,q}^\alpha$.

In the formula (3) that characterizes $\partial b_{p,q}^\alpha$, only A_+B and BA_- give rise to terms that do not correspond to cells of lower dimension. Recall that a total ordering $\iota : \{1, \dots, n\} \rightarrow L(e_\alpha^1)$ is used to form the matrices A_\pm and B , so that the A_+B term corresponds to a sum $\sum_{i < k < j} x_{i,k} b_{\iota(k), \iota(j)}$ in $\partial b_{\iota(i), \iota(j)}$ where $A_+ = (x_{i,j})$. Since $i < k < j$, $b_{\iota(k), \iota(j)} \prec_{\mathcal{A}} b_{\iota(i), \iota(j)}$ unless the sheets $S_{\iota(i)}$ and $S_{\iota(k)}$ are incomparable in $L(e_\alpha^1)$. However, if this is the case then the entry $x_{i,k}$ was checked to be 0 in Lemma 3.2 and Lemma 3.6. The BA_- term is handled similarly.

For a generator $c_{i,j}$ corresponding to a 2-cell, only the terms $A_{v_1}C$ and CA_{v_0} are relevant. Since the sheets above a 2-cell are totally ordered, we have $c_{i,k}, c_{k,j} \prec_{\mathcal{A}} c_{i,j}$ for $i < k < j$. \square

4.2. Elementary modifications to a compatible polygonal decomposition. To prove Theorem 4.1, we begin by showing that the stable tame isomorphism type of (\mathcal{A}, ∂) is invariant under certain local changes to the defining data.

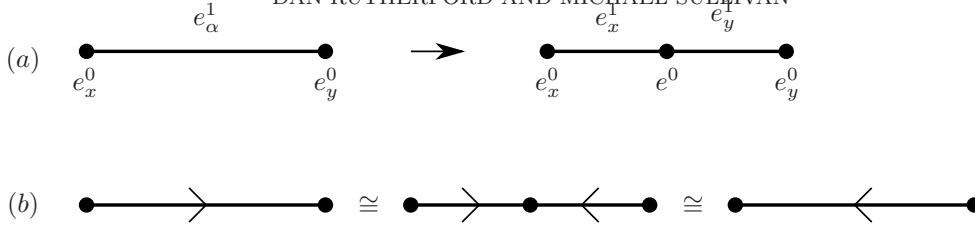


FIGURE 7. (a) Subdividing a 1-cell. (b) Reversing the orientation of a 1-cell.

4.2.1. *Subdividing a 1-cell.* We say that compatible cell decompositions \mathcal{E} and \mathcal{E}' for L are related by **subdividing a 1-cell** if \mathcal{E}' is obtained from \mathcal{E} by dividing a 1-cell, e_α^1 , into two pieces, e_x^1 and e_y^1 , by placing a new vertex e^0 somewhere along e_α^1 . Let e_x^0 and e_y^0 denote the endpoints of e_α^1 in \mathcal{E} that become endpoints of e_x^1 and e_y^1 respectively in \mathcal{E}' . See Figure 7 (a).

Theorem 4.2. *Suppose that \mathcal{E} and \mathcal{E}' are related by subdividing a 1-cell. Moreover assume that:*

- (1) *The orientation of e_x^1 agrees with the orientation of e_α^1 .*
- (2) *The S and T regions at swallow tail points, as well as initial and terminal vertices for 2-cells of \mathcal{E} and \mathcal{E}' are chosen in an identical way.*

Then, the corresponding cellular DGAs are stable tame isomorphic.

Proof. Let (\mathcal{A}, ∂) and $(\mathcal{A}', \partial')$ denote the DGAs associated to \mathcal{E} and \mathcal{E}' respectively. We fix a total ordering of the sheets in $L(e_\alpha^1)$, and use this choice to produce total orderings of $L(e_x^1)$, $L(e_y^1)$, $L(e^0)$, $L(e_x^0)$ and $L(e_y^0)$. We can then collect corresponding generators of \mathcal{A}' (resp. \mathcal{A}) into matrices B_x, B_y, A_0, A_x, A_y (resp. B, A_x, A_y) where, in the case that e_x^0 or e_y^0 is a swallow tail point, we follow the instructions from 3.12 when forming A_x or A_y .

Assume the orientation of e_α^1 is from e_x^0 to e_y^0 , as the argument for the reverse orientation is similar. Then, we have

$$\begin{aligned}\partial B &= A_y(I + B) + (I + B)A_x; \\ \partial' B_x &= A_0(I + B_x) + (I + B_x)A_x;\end{aligned}$$

and

$$\partial' B_y = \begin{cases} A_y(I + B_y) + (I + B_y)A_0 & \text{if } e_y^0 \text{ and } e_\alpha^0 \text{ have the same orientation,} \\ A_0(I + B_y) + (I + B_y)A_y & \text{else.} \end{cases}$$

We can extend the partial ordering, $\prec_{\mathcal{A}'}$, to a total ordering so that all of the generators corresponding to e^0 are greater than the generators corresponding to e_y^0 . Working in increasing order of the $b_{p,q}^y$, we then apply Theorem 2.1 inductively to cancel the generators $b_{p,q}^y$ in pairs with the $a_{p,q}^0$. [Note that the sheets above e^0 are in bijection with the sheets of e_y^1 and have the same partial ordering. Therefore, the generators $b_{p,q}^y$ and $a_{p,q}^0$ are in bijective correspondence.] In the resulting quotient, the entries of A_0 will all be replaced by the corresponding entries of A_y . This is because of the order that we cancel the $b_{p,q}^y$; at the inductive step, any of the B_y terms that would appear in $\partial b_{p,q}^y$ have already been cancelled, and we indeed have $\partial b_{p,q}^y = a_{p,q}^0 + w$ where w is the corresponding entry in A_y (which is less than $a_{p,q}^0$ as required in Theorem 2.1).

Thus, the resulting stable tame isomorphic quotient is obtained from $(\mathcal{A}', \partial')$ by replacing B_y with 0, and replacing all occurrences of A_0 in ∂' with A_y . [Note that this includes in ∂C if C is a 2-cell bordering e_α^1 . When compared with ∂C there is an extra $(I + B_i)^n$ factor in $\partial' C$ corresponding to the edge e_y^1 . This term is replaced with I^n in the quotient.] □

Corollary 4.1. *The stable tame isomorphism type of (\mathcal{A}, ∂) is independent of the orientation of 1-cells of \mathcal{E} .*

Proof. To reverse the orientation of a 1-cell, e_α^1 , of \mathcal{E} , apply Theorem 4.2 twice. First, subdivide into e_x^1 and e_y^1 so that e_y^1 has opposite orientation to e_α^1 . This is just as well a subdivision of \mathcal{E} with the orientation of e_α^1 reversed if the roles of e_x^1 and e_y^1 are interchanged. See Figure 7 (b). □

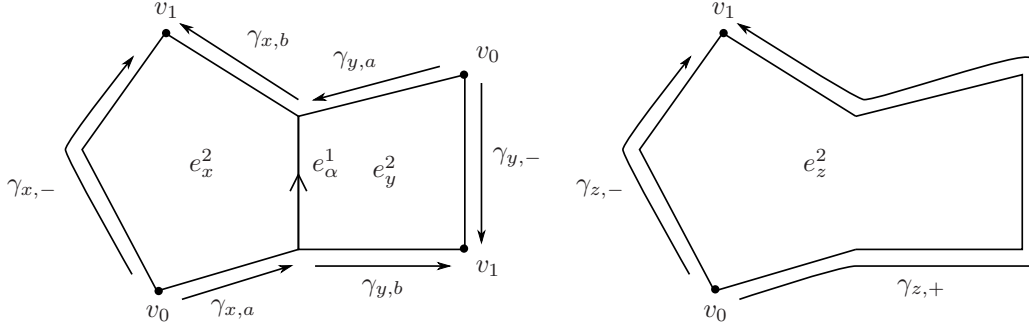


FIGURE 8. The cell decompositions \mathcal{E} (left) and \mathcal{E}' (right). Note that the exponent $\nu = -1$.

4.2.2. *Subdividing a 2-cell.* Suppose that \mathcal{E} and \mathcal{E}' are cell decompositions of S that are compatible with L and that \mathcal{E}' is obtained from \mathcal{E} by removing a 1-cell, e_α^1 , that borders two distinct 2-cells, e_x^2 and e_y^2 , of \mathcal{E} . Thus, \mathcal{E}' has a single 2-cell, e_z^2 , satisfying

$$e_z^2 = e_x^2 \sqcup e_\alpha^1 \sqcup e_y^2.$$

Note that (because \mathcal{E}' is compatible with L) it is necessarily the case that e_α^1 is disjoint from Σ . We say that \mathcal{E} and \mathcal{E}' are related by **subdividing a 2-cell** of \mathcal{E}' . See Figure 8.

Theorem 4.3. *Suppose \mathcal{E} and \mathcal{E}' are related by subdividing a 2-cell. Suppose in addition that:*

- (1) *Initial and terminal vertices are chosen for 2-cells of \mathcal{E} and \mathcal{E}' so that the initial and terminal vertices of e_z^2 are chosen to coincide with those of e_x^2 , and the choice made for all remaining 2-cells of \mathcal{E}' coincides with the choice for the corresponding 2-cells of \mathcal{E} .*
- (2) *The S and T sides of the crossing locus are assigned near each swallowtail point in an identical manner for \mathcal{E} and \mathcal{E}' . Here, we allow for the possibility that e_α^1 borders S and T regions at either of its endpoints.*

Then, the associated cellular DGAs are stable tame isomorphic.

Proof. As in Section 3.13, let Q_x , Q_y , and Q_z denote the polygons associated to the 2-cells e_x^2 , e_y^2 of \mathcal{E} and the 2-cell e_z^2 respectively. In addition, for $w \in \{x, y, z\}$, let $\gamma_{w,\pm}$ denote the paths around ∂Q_w from the initial vertex to the terminal vertex. Note that the edge e_α^1 appears precisely once along either $\gamma_{x,+}$ or $\gamma_{x,-}$ and once along either $\gamma_{y,+}$ or $\gamma_{y,-}$. Since interchanging the notations of γ_+ and γ_- has no effect on the definition of the differential from (15), we can assume without loss of generality that $\gamma_{x,+}$ and $\gamma_{y,+}$ each contain e_α^1 exactly once. Moreover, in view of Corollary 4.1 we may assume that the orientation of e_α^1 agrees with the orientation of $\gamma_{x,+}$. Using $*$ for concatenation of paths, we can then write

$$\gamma_{x,+} = \gamma_{x,a} * e_\alpha^1 * \gamma_{x,b} \quad \text{and} \quad \gamma_{y,+} = \gamma_{y,a} * (e_\alpha^1)^\nu * \gamma_{y,b}$$

where $\nu = \pm 1$ and some of these paths may be constant. Note that

$$(17) \quad \gamma_{z,-} = \gamma_{x,-} \quad \text{and} \quad \gamma_{z,+} = \gamma_{x,a} * ((\gamma_{y,a})^{-1} * \gamma_{y,-} * (\gamma_{y,b})^{-1})^\nu * \gamma_{x,b}.$$

See Figure 8. In view of Corollary 4.1, we may assume the orientation of e_α^1 is from the initial vertex to the terminal vertex. Using Theorem 2.1, we now show that the DGAs (\mathcal{A}, ∂) and (\mathcal{A}, ∂') are stable tame isomorphic. Collect generators of \mathcal{A} (resp. \mathcal{A}') associated to e_x^2 and e_y^2 (resp. to e_z^2) into matrices C_x and C_y (resp. C_z). We choose an extension of the ordering of generators of \mathcal{A} from $\prec_{\mathcal{A}}$ to a total order so that the generators corresponding to e_α^1 are larger than any other 1-cell generators. Using this ordering we can inductively cancel the $c_{i,j}^y$ with the $b_{i,j}^\alpha$ using Theorem 2.1. [Indeed, expanding the product of edges along $\gamma_{y,+}$ that appears in equation (15) for ∂C_y allows us to write

$$(18) \quad \partial C_y = A_{v_1} C_y + C_y A_{v_0} + B_\alpha + X$$

where the matrix X is strictly upper triangular with and has its i, j entry in the sub-algebra generated by generators from 0-cells and 1-cells different from e_α^1 and also those $b_{i',j'}^\alpha$ with $i \leq i' < j' \leq j$ such that at least one of the first and last inequalities is strict, i.e. those $b_{i',j'}^\alpha$ with $b_{i',j'}^\alpha \prec_{\mathcal{A}} b_{i,j}^\alpha$. Therefore, we can apply Theorem 2.1 and inductively quotient by ideals generated by $c_{i,j}$ and $\partial c_{i,j}$ according to the increasing ordering of these generators. For the inductive step, it is important to observe that, as

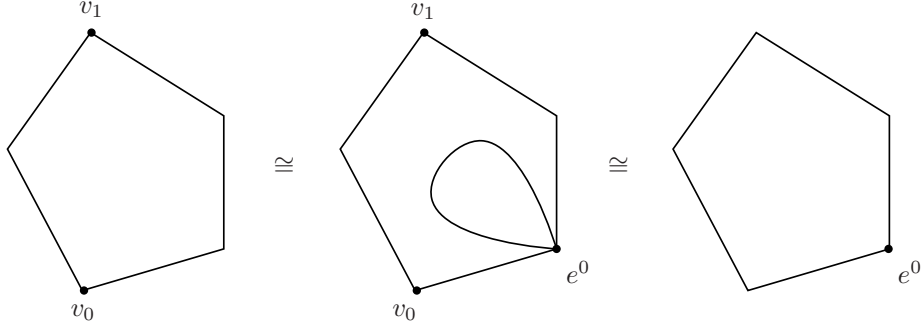


FIGURE 9. The cell decompositions and initial and terminal vertices used in the proof of Corollary 4.2.

long as $c_{i',j'} = 0$ for i', j' such that $c_{i',j'} \prec_A c_{i,j}$, equation (18) shows that $\partial c_{i,j} = b_{i,j}^\alpha + w$ where w belongs to the subalgebra generated by those x with $x \prec_A b_{i,j}^\alpha$.

The resulting quotient has the $c_{i,j}^y$ and $b_{i,j}^\alpha$ removed from the generating set. Moreover, the relations

$$\begin{aligned}
 0 &= C_y, \quad \text{and} \\
 0 &= \partial C_y = A_{v_1} C_y + C_y A_{v_0} + \underbrace{Y_j^{\eta_j} \cdots Y_1^{\eta_1}}_{\gamma_{y,+}} + \underbrace{Y_m^{\eta_m} \cdots Y_{j+1}^{\eta_{j+1}}}_{\gamma_{y,-}} = \\
 &\quad \underbrace{Y_j^{\eta_j} \cdots Y_{l+1}^{\eta_{l+1}}}_{\gamma_{y,b}} (I + B_\alpha)^\nu \underbrace{Y_{l-1}^{\eta_{l-1}} \cdots Y_1^{\eta_1}}_{\gamma_{y,a}} + \underbrace{Y_m^{\eta_m} \cdots Y_{j+1}^{\eta_{j+1}}}_{\gamma_{y,-}}
 \end{aligned}$$

allow us to find

$$(19) \quad (I + B_\alpha) = \underbrace{([Y_j^{\eta_j} \cdots Y_{l+1}^{\eta_{l+1}}]^{-1})}_{\gamma_{y,b}} \underbrace{Y_m^{\eta_m} \cdots Y_{j+1}^{\eta_{j+1}}}_{\gamma_{y,-}} \underbrace{[Y_{l-1}^{\eta_{l-1}} \cdots Y_1^{\eta_1}]^{-1}}_{\gamma_{y,a}}^\nu.$$

(We have indicated which parts of the paths $\gamma_{y,-}$ and $\gamma_{y,+}$ the various portions of the product correspond to. The Y_i matrices are as in Section 3.13, and some of them may be S or T matrices from corners decorated at swallowtail points.)

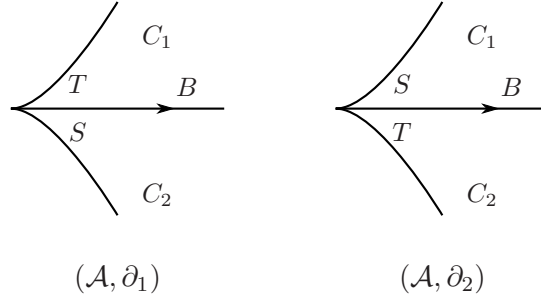
Finally, note that ∂C^x becomes identical to $\partial' C^z$ after making the substitution (19). [Compare with (17).] It follows that the identification of C_x with C_z provides an isomorphism between this quotient of (\mathcal{A}, ∂) and $(\mathcal{A}', \partial')$. □

Corollary 4.2. *The stable tame isomorphism type of (\mathcal{A}, ∂) is independent of the choices of initial and terminal vertices for 2-cells.*

Proof. Let \mathcal{E} be an L -compatible polygonal decomposition of S , and let e_α^2 be any 2-cell of \mathcal{E} . Let (\mathcal{A}, ∂) denote a cellular DGA formed from using v_0 and v_1 as the initial and terminal vertices for e_α^2 .

Pick a 0-cell, e^0 , appearing in the boundary of e_α^2 and modify the cell decomposition \mathcal{E} to \mathcal{E}' by adding a loop edge, e^1 , from e^0 to itself that is contained in e_α^2 . This subdivides e_α^2 into two pieces e_x^2 and e_y^2 where e_x^2 is exterior to e^1 and e_y^2 is interior to e^1 . Choose initial and terminal vertices of e_x^2 to be v_0 and v_1 , and choose both initial and terminal vertices of e_y^2 to be e^0 . Keep all other choices the same. Then according to Theorem 4.3 the DGA $(\mathcal{A}', \partial')$ associated to \mathcal{E}' is stable tame isomorphic to (\mathcal{A}, ∂) . On the other hand, if we reverse the role of e_x^2 and e_y^2 , we see that the DGA associated to \mathcal{E} modified so that the initial and terminal vertices v_0 and v_1 for e_α^2 are both replaced with e^0 is also stable tame isomorphic to $(\mathcal{A}', \partial')$. Since stable tame isomorphism satisfies the properties of an equivalence relation, it follows that the stable tame isomorphism type of (\mathcal{A}, ∂) is independent of the choice of v_0 and v_1 . See Figure 9. □

4.2.3. Deleting an edge with 1-valent vertex. Suppose that an edge of \mathcal{E} has an endpoint at a 1-valent vertex. The edge and vertex must be disjoint from the singular set $\Sigma \subset S$, and therefore we can delete them to produce another L -compatible cell decomposition of S which we denote as \mathcal{E}' .

FIGURE 10. Generators and decorations near the swallow tail point s .

Theorem 4.4. *For \mathcal{E} and \mathcal{E}' related by deleting an edge with 1-valent vertex, the associated cellular DGAs for L are stable tame isomorphic.*

Proof. Cancel the generators of the 1-cell with the generators of the univalent vertex as in the proof of Theorem 4.2. \square

4.3. Independence of S and T decorations at swallow tail points.

Theorem 4.5. *The stable tame isomorphism type of the cellular DGA is independent of the choice of S and T regions at swallowtail points.*

Proof. It suffices to establish stable tame isomorphism between cellular DGAs $(\mathcal{A}, \partial_1)$ and $(\mathcal{A}, \partial_2)$ arising from a common L -compatible decomposition, \mathcal{E} , such that the choice of S and T regions is opposite at a single swallowtail point, $s \in \Sigma$, as pictured in Figure 10. We give the proof only in the case of an upward swallowtail with n sheets above the swallow tail region and so that sheets $k, k+1$, and $k+2$ meet at the swallowtail point. The case of a downward swallowtail is similar.

Using Theorems 4.2 and 4.3 as well as Corollaries 4.1 and 4.2, we may assume that

- The 1-cell, e_α^1 , that has an endpoint at s and is contained in the crossing arc is oriented away from s .
- The 2-cells containing the S and T regions are distinct.
- The initial and terminal vertices of these 2-cells are disjoint from s , and the orientations of the paths γ_\pm around the boundaries of the 2-cells agree with the orientation of B as they pass around the corner of the S and T regions.

Fixing an orientation of S (the base surface) near the swallowtail point, we collect the generators for the 2-cell that sits to the left (resp. right) of e_α^1 into matrices C_1 and C_2 . We form a matrix B from the generators for e_α^1 by using the total ordering of sheets as it appears to the left of e_α^1 , so that rows and columns of B and C_1 correspond to the same sheets. The generators associated to the swallowtail point itself are placed in an $(n-2) \times (n-2)$ matrix A , and we have $n \times n$ matrices A_S , A_T , S , and T as defined in Section 3.11. We suppose that for $(\mathcal{A}, \partial_1)$ (resp. $(\mathcal{A}, \partial_2)$), the S region is to the right (resp. left) of e_α^1 and the T region is to the left (resp. right) of e_α^1 . See Figure 10.

We define an algebra morphism $\psi : (\mathcal{A}, \partial_1) \rightarrow (\mathcal{A}, \partial_2)$ by requiring that when applied entry by entry

$$(20) \quad \psi(I + B) = (I + B)(ST),$$

and $\psi(x) = x$ for any generator that is not associated to the 1-cell e_α^1 . To verify that (20) can be obtained by a unique assignment of values $\psi(b_{i,j}^\alpha)$ it is necessary to note that $(I + B)(ST) = I + X$ where X is strictly upper-triangular with $k+1, k+2$ entry is equal to 0. The latter claim holds since it is true for B and also for

$$(21) \quad ST = TS = I + \hat{A}_{k,k+1} E_{k+2,k}.$$

We claim that ψ is in fact a tame isomorphism from \mathcal{A} to itself. To verify, note that for each generator $b_{i,j}^\alpha$, since the $\hat{A}_{k,k+1} E_{k+2,k}$ term from (21) is strictly upper triangular, it follows from (20) that

$$\psi(b_{i,j}^\alpha) = b_{i,j}^\alpha + w_{i,j}$$

where $w_{i,j}$ belongs to the sub-algebra generated by generators x with $x \prec_A b_{i,j}^\alpha$. Define $\psi_{i,j} : \mathcal{A} \rightarrow \mathcal{A}$ so that $\psi_{i,j}(b_{i,j}^\alpha) = b_{i,j}^\alpha + w_{i,j}$ and $\psi_{i,j}(x) = x$ for any generator not equal to $b_{i,j}^\alpha$. The $\psi_{i,j}$ are elementary isomorphisms¹. Moreover, it is straightforward to check that ψ can be written as the composition of all of the $\psi_{i,j}$ provided that we compose in such a way that the subscripts increase from right to left, with respect to a total ordering of the $b_{i,j}^\alpha$ that extends \prec_A . Thus, ψ is indeed a tame isomorphism.

To complete the proof, we check that $\psi \circ \partial_1 = \partial_2 \circ \psi$. Note that it is enough to verify this equality when the two sides are applied entry-by-entry to the matrices C_1 , C_2 , and $I + B$. This is because these matrices contain the only generators for which the entries of B can appear in the differential. To this end, we compute:

1.

$$\begin{aligned} \psi \circ \partial_1(C_1) &= \psi(\cdots (I + B)T \cdots + \cdots) = \\ &\cdots (I + B)(ST)T \cdots + \cdots = \cdots (I + B)S \cdots + \cdots = \partial_2 \circ \psi(C_1) \end{aligned}$$

where we used that T is self inverse.

2. Using Q for the permutation matrix of the transposition $(k + 1 \ k + 2)$,

$$\begin{aligned} \psi \circ \partial_1(C_2) &= \psi(\cdots Q(I + B)QS \cdots + \cdots) = \\ &\cdots Q(I + B)(TS)QS \cdots + \cdots = \cdots Q(I + B)Q(TS)S \cdots + \cdots = \\ &\cdots Q(I + B)QT \cdots + \cdots = \partial_2 \circ \psi(C_1) \end{aligned}$$

where we used (21), that $(TS)Q = Q(TS)$ since the $k + 1$ and $k + 2$ columns and rows of TS agree with those columns of the identity matrix, and that $S^2 = I$. (The Q 's appear in $\partial_i(C_2)$ because the matrix B was formed using the ordering of sheets above C_1 which has the $k + 1$ and $k + 2$ sheets in the opposite order that they appear in above C_2 .)

3. With A_+ denoting a matrix corresponding to the terminal end point of e_α^1 , we compute

$$\begin{aligned} \psi \circ \partial_1(I + B) &= \psi[A_+(I + B) + (I + B)A_T] = \\ &A_+(I + B)ST + (I + B)STA_T = A_+(I + B)ST + (I + B)S\hat{A}_{k,k+1}T = \\ &A_+(I + B)ST + (I + B)(A_S S + \partial_2 S)T = \\ &[A_+(I + B) + (I + B)A_S]ST + (I + B)(\partial_2 S)T = \\ &\partial_2(I + B) \cdot ST + (I + B) \cdot \partial_2(ST) = \partial_2 \circ \psi(I + B). \end{aligned}$$

Here, the 3rd and 4th equalities used identities from Lemma 3.4. □

4.4. Common refinements for L -compatible cell decompositions. Let \mathcal{E}_1 and \mathcal{E}_2 be any cell decompositions for S that are compatible with L . After modifying \mathcal{E}_2 by an ambient isotopy of S , preserving the sets Σ_2 and Σ_1 , we can assume that the cells of \mathcal{E}_1 and \mathcal{E}_2 intersect transversally in each of the strata Σ_2 , Σ_1 , and $\Sigma_0 := S \setminus \Sigma$. That is, if $e_\alpha^d \in \mathcal{E}_1$ and $e_\beta^{d'} \in \mathcal{E}_2$ satisfy $e_\alpha^d, e_\beta^{d'} \subset \Sigma_i$, then they intersect transversally when viewed as subsets of the $2 - i$ -dimensional manifold Σ_i . [This just amounts to requiring that the only common 0-cells of \mathcal{E}_1 and \mathcal{E}_2 are in Σ_2 , and that all 1-cells and 0-cells of \mathcal{E}_1 and \mathcal{E}_2 are transverse in $S \setminus \Sigma$.] Such a modification of \mathcal{E}_2 does not affect the cellular DGA.

Theorem 4.6. *With the above transversality assumption, we can transform \mathcal{E}_1 into \mathcal{E}_2 via a sequence of the local modifications appearing in Theorems 4.2, 4.3, and 4.4.*

Theorem 4.6, together with Corollaries 4.1, 4.2, and Theorem 4.5, establish Theorem 4.1

Proof. By subdividing edges and 2-cells, we can assume that:

$$(22) \quad \text{All characteristic maps for both } \mathcal{E}_i \text{ are embeddings of closed disks into } S.$$

We construct a 3-rd L -compatible cell decomposition \mathcal{E} , and then show that \mathcal{E} is related to both \mathcal{E}_1 and \mathcal{E}_2 in the required manner.

¹In fact, the $\psi_{i,j}$ satisfy the stronger requirement discussed in Remark 2.2.

Defining \mathcal{E} . We start by defining \mathcal{E} on Σ . Here, the 0-cells of \mathcal{E} are the union of the 0-cells of \mathcal{E}_1 and \mathcal{E}_2 that belong to Σ . These two sets are disjoint except for Σ_2 , and to obtain the one cells of \mathcal{E} in Σ we just subdivide the 1-cells of \mathcal{E}_1 at any 0-cells of \mathcal{E}_2 not in Σ_2 .

To extend the construction of \mathcal{E} to all of Σ , we include the 1-skeletons of both \mathcal{E}_1 and \mathcal{E}_2 in the 1-skeleton of \mathcal{E} . Note, that all 1-cells and 0-cells in $S \setminus \Sigma$ intersect transversally, so we triangulate this union of 1-skeletons by adding new 4-valent vertices at intersections of 1-cells of \mathcal{E}_1 and \mathcal{E}_2 in $S \setminus \Sigma$. Denote this union of 1-skeletons as X .

The components of $S \setminus X$ are open surfaces with polygonal boundary in X . They are planar, since they are contained in cells of the \mathcal{E}_i , and can be subdivided into disks by adding some extra 1-cells that connect distinct boundary components. This completes the construction of \mathcal{E} .

It remains to prove that \mathcal{E} is related to both of the \mathcal{E}_i by the 3 allowable moves:

- (A) Adding/deleting an edge with totally ordered sheets that borders two distinct 2-cells.
- (B) Adding/deleting a 0-cell that subdivides a 1-cell.
- (C) Adding/deleting an edge with a 1-valent vertex.

For this purpose, let e be a 2-cell in \mathcal{E}_1 and consider the polygonal decomposition of D^2 consisting of preimages of cells of \mathcal{E} .

Step 1. Remove all 0 and 1-cells from the interior of e , by repeated application of (A)-(C). This is done as follows. If there is more than one 2-cell in e , then there must be some edge that borders two distinct 2-cells which we delete to decrease the number of 2-cells. [The sheets of L are totally ordered above the edge since it is in the interior of e and the singular set Σ is contained in the 1-skeleton of \mathcal{E}_1 .] Once there is only one 2-cell in the polygonal decomposition of $e = D^2$, then either the 1-skeleton is the boundary of D^2 or there exists a 1-valent vertex in the interior. [To find it, start with any edge in the interior. Building a path inductively starting with this edge, we can either find some closed loop in the 1-skeleton that contains this edge, which would contradict their only being one 2-cell, or we can find a 1-valent vertex.] Repeatedly cancelling 1-valent vertices with their corresponding edges removes all remaining 0 and 1-cells from the interior of D^2 .

Step 2. Applying Step 1 to each of the 2-cells of \mathcal{E}_1 leaves an L -compatible polygonal decomposition that has the same 2-cells and same 1-skeleton as \mathcal{E}_1 . The only remaining difference is that some of the 1-cells of \mathcal{E}_1 are subdivided, and we can remove these subdivisions using (A). □

5. EXAMPLES AND EXTENSIONS

In Sections 5.1 and 5.2, we compute some examples of the cellular DGAs of Legendrian spheres. In Sections 5.3 and 5.4, we extend the definition of the DGA to allow for Legendrians with (non-generic) cone point singularities, and to allow the flexibility of having more than one crossing arc above a given 1-cell. We then apply this extensions to give an algebraic description of the DGA of surfaces obtained from spinning a 1-dimensional Legendrian around an axis, allowing for the possibility that the axis intersects the Legendrian. Finally, in Section 5.6 we compute the DGA of a family of Legendrian spheres. Many pairs of spheres from this family have linearized contact homology groups with the same ranks but can be distinguished with product operations.

For some of these examples, the LCH has been (partially or completely) computed before with holomorphic curves consistent with our computations. So such computations can be viewed as “empirical evidence” that the cellular DGA is the same as LCH.

5.1. Legendrian spheres. For these next two examples, the linearized contact homology has already been partially computed and used to show that they form an infinite family of distinct Legendrian spheres with the same rotation class and Thurston-Bennequin invariant [15, Proposition 4.10 and Theorem 4.11].

Example 5.1. Recall the Legendrian L_1 pictured in Figure 3. We use the polygonal decomposition, \mathcal{E} , of $\pi_x(L_1)$ indicated in Figure 11.

Generators for the cellular DGA are determined once we assign an indexing set to the sheets above each cell of \mathcal{E} . In all cases, if $L(e_\alpha^i)$ consists of n sheets, we use $\{1, \dots, n\}$ for the indexing set in such a way that the z -coordinates are non-increasing, $z(S_i) \geq z(S_{i+1})$. This uniquely specifies the indexing for cells that are not contained in the crossing locus. For such cells, it is convenient to specify

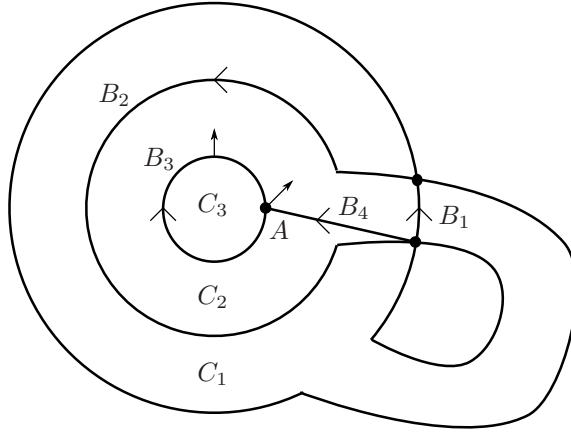


FIGURE 11. To arrive at a polygonal decomposition of $\pi_x(L_1)$ that is compatible with L_1 we take the 0-cells to consist of the codimension 2 singularities of the base projection together with a point on the crossing locus. The 1-cells are the resulting pieces of the cusp and crossing locus together with an additional 1-cell that subdivides the annular component of $\pi_x(L_1) \setminus \Sigma$ into a disk. Orientations are assigned to 1-cells as pictured. The arrows pointing out of A and B_3 indicate a preferred ordering of sheets at these cells.

the indexing of sheets by choosing a bordering 2-cell, and ordering sheets as they appear above this 2-cell. In Figure 11 the choice is indicated with arrows pointing from the cells in the crossing locus to a neighboring 2-cell.

In the definition of the cellular DGA, the generators associated to a given cell are placed into several, possibly distinct matrices depending on the context, eg. generators for a 1-cell may be placed into different matrices when computing the differentials of the two bordering 2-cells. When working through examples, it is convenient to fix particular matrices containing the generators of each cell and then write all differentials in terms of these initial matrices. For L_1 , we form initial 2×2 matrices B_1, B_2 , and C_1 , as well as 4×4 matrices A, B_3, B_4, C_2 and C_3 by ordering rows and columns according to our indexing of sheets. All of these matrices are strictly upper triangular. We note that the matrices A and B_3 have their $(2,3)$ -entry equal to 0 and that there are no generators associated to cells that appear on the boundary of $\pi_x(L_1)$. We choose as $v_1 = v_0$ for both C_1 and C_2 the initial point of B_4 .

Differentials are then determined by the matrix formulas

$$\begin{aligned}
 (23) \quad & \partial A = A^2; \\
 & \partial B_1 = N(I + B_1) + (I + B_1)N = 0; \\
 & \partial B_2 = N(I + B_2) + (I + B_2)N = 0; \\
 & \partial C_1 = NC_1 + C_1N + (I + B_2)(I + B_1) + I = \begin{bmatrix} 0 & b_{1,2}^1 + b_{1,2}^2 \\ 0 & 0 \end{bmatrix}; \\
 & \partial B_3 = A(I + B_3) + (I + B_3)A; \\
 & \partial B_4 = A(I + B_4) + (I + B_4)\hat{N}_{3,4}; \\
 & \partial C_2 = \hat{N}_{3,4}C_2 + C_2\hat{N}_{3,4} + (I + B_4)^{-1}(I + B_3)(I + B_4)(I + \tilde{B}_{2,(1,2)})(I + \tilde{B}_{1,(3,4)}) + I; \\
 & \partial C_3 = (Q A Q)C_3 + C_3(Q A Q) + (I + Q B_3 Q) + I.
 \end{aligned}$$

Here, the notation $\hat{X}_{k,l}$ (resp. $\tilde{X}_{k,l}$) indicates the matrix obtained from X by inserting the 2×2 block N (resp. 0) along the diagonal at row and column k and l . In particular, $\tilde{B}_{j,(k,l)} = \left(\tilde{B}_j\right)_{k,l}$. In addition,

$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is the permutation matrix associated to the transposition $(2\ 3)$. Note that A and B_3 are conjugated by Q in the formula for ∂C_3 because the indexing of sheets above A and B_3 was

chosen to agree with the ordering above C_2 where sheets 2 and 3 appear in opposite order than they do above C_3 .

The cellular DGA of L_1 has 31 generators. While the differential of each generator is easily obtained from the above matrix formulas, calculations with this version of the DGA would probably be handled best by a computer. However, applying Theorem 2.1 allows us to find a much smaller stable tame isomorphic quotient. We essentially cancel matrices of generators in pairs, following a sequence of simple homotopy equivalences applied to the polygonal decomposition of $\pi_x(L_1)$.

Proposition 5.1. *The DGA of L_1 is stable tame isomorphic to a DGA, $(\mathcal{A}_1, \partial)$, with 3 generators x, y and z of degrees*

$$|x| = -1; \quad |y| = |z| = 2$$

with differentials

$$\partial x = \partial y = 0; \quad \partial z = xy + yx.$$

Proof. We will show how to cancel generators of the cellular DGA (\mathcal{A}, ∂) using Theorem 2.1 in order to arrive at $(\mathcal{A}_1, \partial)$ as a stable tame isomorphic quotient. Throughout we use the same notation for generators and their equivalence classes in various quotients of (\mathcal{A}, ∂) , and we use the symbol \doteq to indicate that equality holds in the currently considered quotient.

Consider the equation

$$\partial \boxed{B_4} = \boxed{A} + AB_4 + (I + B_4)\widehat{N}_{3,4}.$$

The $(2, 3)$ -entries of A and $\widehat{N}_{3,4}$ are both 0, so, using the notation $x := b_{2,3}^4$, we have $\partial x = 0$. The remaining entries of B_4 can be inductively canceled with the entries of A , leaving

$$(24) \quad B_4 \doteq xE_{2,3}; \quad A \doteq (I + xE_{2,3})\widehat{N}_{3,4}(I + xE_{2,3}) = \begin{bmatrix} 0 & 1 & x & 0 \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Proceeding in a similar manner, the equation

$$(25) \quad \partial \boxed{C_3} = (QAQ)C_3 + C_3(QAQ) + Q\boxed{B_3}Q$$

allows us to cancel all entries of C_3 , except for $y := c_{2,3}^3$, with the corresponding entries of QB_3Q . In the quotient, we have

$$C_3 \doteq yE_{2,3}; \quad B_3 \doteq AQ(yE_{2,3})Q + Q(yE_{2,3})QA \doteq \begin{bmatrix} 0 & xy & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & yx \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The equation for ∂C_2 becomes

$$(26) \quad \partial C_2 \doteq \begin{bmatrix} 0 & 0 & c_{2,3}^2 & c_{1,3}^2 + c_{2,4}^2 \\ 0 & 0 & 0 & c_{2,3}^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & xy & yxy & 0 \\ 0 & 1 & 0 & yxy \\ 0 & 0 & 1 & yx \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b_{1,2}^1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & b_{1,2}^2 \\ 0 & 0 & 0 & 1 \end{bmatrix} + I.$$

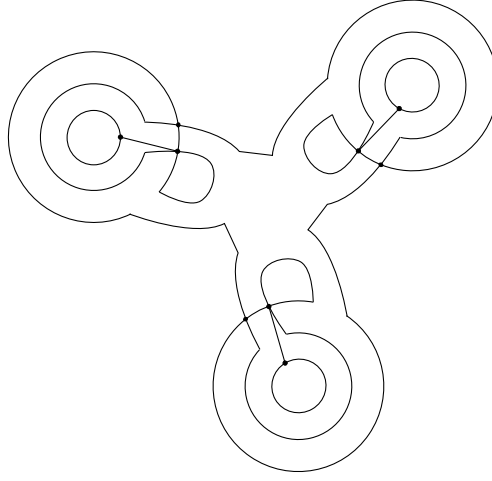
We can thus cancel as indicated in the following

$$\partial \boxed{c_{1,2}^2} \doteq xy + \boxed{b_{1,2}^1}, \quad \partial \boxed{c_{1,3}^2} \doteq \boxed{c_{2,3}^2} + \cdots, \quad \partial \boxed{c_{1,4}^2} \doteq \boxed{c_{2,4}^2} + \cdots, \quad \partial \boxed{c_{3,4}^2} \doteq yx + \boxed{b_{1,2}^2}.$$

The first and last equations imply $b_{1,2}^1 \doteq xy$ and $b_{1,2}^2 \doteq yx$. Finally,

$$(27) \quad \partial c_{1,2}^1 = b_{1,2}^1 + b_{1,2}^2.$$

We have canceled all generators except for x, y and $z := c_{1,2}^1$. Since $\partial x = 0 = \partial y$, the result follows. \square

FIGURE 12. The Legendrian L_k with $k = 3$.

Example 5.2. Let L_k denote k copies of L_1 from Example 5.1 cusp-connect summed as in Figure 12. Each cell $A, B_1, B_2, B_3, B_4, C_2, C_3$ in Example 5.1 now has k copies which we indicate with a superscript, such as B_1^1, \dots, B_1^k . Note that there is only one C_1 cell. We add a corresponding second superscript when labeling the Reeb chords; for example, the unique non-zero entry in B_2^j is $b_{1,2}^{2,j}$. Except for ∂C_1 , the differentials are exactly as in (23) with appropriate subscripts. For example, $\partial B_4^j = A^j(I + B_4^j) + (I + B_4^j)\widehat{N}_{3,4}$. For C_1 , if we set $v_0 = v_1$ equal to the start point of cell B_4^1 , then

$$\partial C_1 = \widehat{N}_{3,4}C_1 + C_1\widehat{N}_{3,4} + (1 + B_2^1)(1 + B_1^k)(1 + B_2^2) \cdots (1 + B_1^2)(1 + B_2^2)(1 + B_1^1) + I.$$

Proposition 5.2. *The DGA of L_k is stable tame isomorphic to a DGA, $(\mathcal{A}_k, \partial)$, with $2k+1$ generators $x_1, y_1, \dots, x_k, y_k, z$ of degrees*

$$|x_j| = -1; \quad |y_j| = |z| = 2$$

with differentials

$$\partial x_j = \partial y_j = 0; \quad \partial z = \sum_{j=1}^k (x_j y_j + y_j x_j).$$

Proof. We do the same cancelations as in (24), (25) and (26), labeling the remaining generators $x_j := b_{2,3}^{4,j}, y_j = c_{2,3}^{3,j}$. Relations in (26) imply $b_{1,2}^{1,j} \doteq x_j y_j$ and $b_{1,2}^{2,j} \doteq y_j x_j$. The k -copy version of (27) is $\partial c_{1,2}^1 = \sum_{j=1}^k (b_{1,2}^{1,j} + b_{1,2}^{2,j})$. Setting $z := c_{1,2}^1$, the result follows. \square

As alluded to before, a partial computation for these examples appears in [15, Proposition 4.10 and Theorem 4.11]. In particular, using J -holomorphic disks, the paper computes that the degree -1 subspace of the linearized Legendrian contact homology is $\text{LinLCH}_{-1}(L_k) = (\mathbb{Z}/2)^k$. [Linearized LCH was introduced in [5]. Briefly, an augmentation, if it exists, is a DGA-morphism $\epsilon : (\mathcal{A}, \partial) \rightarrow (\mathbb{Z}_2, 0)$. Define a graded algebra morphism $\phi_\epsilon : \mathcal{A} \rightarrow \mathcal{A}$ on the generators by $\phi_\epsilon(x) = x + \epsilon(x)$. Define $d_\epsilon(x)$ as the sum of words of length one in $\phi_\epsilon \partial \phi_\epsilon^{-1}(x)$. One can check that $d_\epsilon^2 = 0$. The linearized LCH is $\text{Ker}(d_\epsilon)/\text{Im}(d_\epsilon)$.] This computation agrees with ours. To see this consistency, note that since there are no generators of grading 0, there is a unique (graded) augmentation. Thus, the degree -1 subspace of the linearized cellular homology is freely generated by x_1, \dots, x_k , and so isomorphic to $(\mathbb{Z}_2)^k$.

5.2. An example with swallowtail points. Consider the Legendrian, L , which is pictured in Figure 6 together with a compatible decomposition of $\pi_x(L)$. As in the earlier example, we fix upper-triangular matrices $A_1, A_2, B_1, \dots, B_5, C_1, \dots, C_4$ by placing generators of (\mathcal{A}, ∂) into rows and columns according to the ordering of sheets above each cell. For the 1-cell B_5 that contains the crossing locus, we use the ordering of sheets as they appear above C_4 . (This choice of ordering is indicated in Figure 6 by the arrow pointing from B_5 to C_4 .)

The differential of any generator can be computed using the matrix formulas of Section 3.7. Here we write out ∂C_3 and ∂B_5 explicitly.

To compute ∂C_3 we would consider the polygon Q pictured in Figure 6 that arises from considering the boundary of C_3 along with the location of the S and T decorations. Note that, for L , the matrices S_1, T_1 and S_2, T_2 associated to the swallowtail points A_1 and A_2 are all equal to $I + E_{2,3}$. Taking the upper vertex of Q for v_0 and v_1 , with the path around Q from v_0 to v_1 chosen to be counter-clockwise, we have

$$(28) \quad \partial C_3 = \hat{N}_{3,4} C_3 + C_3 \hat{N}_{3,4} + I + (I + \tilde{B}_{4,(3,4)}) S_2 (I + Q B_5 Q) S_1 (I + \tilde{B}_{2,(1,2)})$$

where Q is the permutation matrix of the transposition $(2, 3)$ which appears because the labeling of sheets S_2 and S_3 above B_5 and C_3 is opposite. (Notations are as in Example 5.1.)

We now consider ∂B_5 . As the ordering of rows and columns of B_5 agrees with the ordering of sheets above the T_1 and T_2 corners at the swallow tail points, we have

$$(29) \quad \partial B_5 = (A_2)_T (I + B_5) + (I + B_5) (A_1)_T,$$

where

$$(A_1)_T = (I + E_{2,3}) \begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & a_{1,2}^1 \\ & & & 0 \end{bmatrix} (I + E_{2,3}); \quad (A_2)_T = (I + E_{2,3}) \begin{bmatrix} 0 & a_{1,2}^2 & & \\ & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} (I + E_{2,3})$$

The cellular DGA (\mathcal{A}, ∂) for L has 25 generators, but once again appropriate applications of Theorem 2.1 produce a much smaller stable tame isomorphic quotient.

Proposition 5.3. *The cellular DGA of L is stable tame isomorphic to a DGA with a single generator x with $|x| = 2$ and $\partial x = 0$.*

Remark 5.3. This agrees with the DGA of the standard 2-dimensional Legendrian unknot, U . In fact, it can be shown that L and U are Legendrian isotopic.

Proof. We first compute

$$\boxed{\partial b_{1,2}^2} = \boxed{a_{1,2}^1} + 1$$

and apply Theorem 2.1 to cancel $b_{1,2}^2$ with $a_{1,2}^1$ so that $b_{1,2}^2 \doteq 0$ and $a_{1,2}^1 \doteq 1$. (Again, \doteq denotes equality in the presently considered quotient.) Independent of the choices of v_0 and v_1 for the 2-cell C_1 , we have

$$\boxed{\partial c_{1,2}^1} = b_{1,2}^1 + b_{1,2}^2 \doteq \boxed{b_{1,2}^1}$$

so that we may cancel $c_{1,2}^1$ with $b_{1,2}^1$.

Next, a parallel sequence of applications of Theorem 2.1 results in

$$c_{1,2}^2 \doteq b_{1,2}^3 \doteq b_{1,2}^4 \doteq 0 \quad \text{and} \quad a_{1,2}^2 \doteq 1.$$

For computing ∂C_4 , we take v_0 and v_1 to be the vertex that is the common endpoint of B_1 and B_3 and choose the path around the boundary of C_4 to be clock-wise. We then have

$$\begin{aligned} \partial C_4 = & \begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} C_4 + C_4 \begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} + I + (I + \tilde{B}_{3,(3,4)}) T_2 (I + B_5) T_1 (I + \tilde{B}_{1,(1,2)}) \doteq \\ & \begin{bmatrix} 0 & 0 & c_{2,3}^4 & c_{24}^4 + c_{1,3}^4 \\ & & c_{2,3}^4 & \\ & & 0 & \\ & & & 0 \end{bmatrix} + I + \begin{bmatrix} 1 & & & \\ & 1 & 1 & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & b_{1,2}^5 & b_{1,3}^5 & b_{14}^5 \\ & 1 & 0 & b_{24}^5 \\ & & 1 & b_{34}^5 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & 1 & \\ & & 1 & \\ & & & 1 \end{bmatrix}. \end{aligned}$$

In particular, we can use the following entry-by-entry evaluations to cancel all of the $c_{i,j}^4$ along with $b_{1,2}^5$ and b_{34}^5 as indicated:

$$\begin{aligned} \boxed{\partial c_{1,2}^4} & \doteq \boxed{b_{1,2}^5}; & \boxed{\partial c_{34}^4} & \doteq \boxed{b_{34}^5}; \\ \boxed{\partial c_{1,3}^4} & \doteq \boxed{c_{2,3}^4} + x_b; & \boxed{\partial c_{14}^4} & \doteq \boxed{c_{24}^4} + c_{1,3}^4 + y_b \doteq \boxed{c_{24}^4} + y_b; \end{aligned}$$

where the terms x_b and y_b are in the subalgebra generated by entries of B_5 . (Hence, x_b and y_b satisfy the condition of the term w from the statement of Theorem 2.1.)

At this point, equation (28) has simplified to

$$\partial C_3 = \begin{bmatrix} 0 & 0 & c_{2,3}^3 & c_{24}^3 + c_{1,3}^3 \\ & & c_{2,3}^3 & \\ & & 0 & \\ & & 0 & \end{bmatrix} + I + \begin{bmatrix} 1 & & & \\ & 1 & 1 & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & b_{1,3}^5 & 0 & b_{14}^5 \\ & 1 & 0 & \\ & & 1 & b_{24}^5 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & 1 & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

We cancel

$$\begin{aligned} \boxed{\partial c_{1,2}^3} &\doteq \boxed{b_{1,3}^5}; & \boxed{\partial c_{34}^3} &\doteq \boxed{b_{24}^5}; \\ \boxed{\partial c_{1,3}^3} &\doteq \boxed{c_{2,3}^3}; & \boxed{\partial c_{14}^3} &\doteq \boxed{c_{24}^3} + b_{14}^5. \end{aligned}$$

The only remaining generator is b_{14}^5 which has $|b_{14}^5| = 2$. In the current quotient, $b_{1,2}^5 \doteq b_{13}^5 \doteq b_{24}^5 \doteq b_{34}^5 \doteq 0$, and $a_{1,2}^1 \doteq a_{1,2}^2 \doteq 1$, and it follows that (29) simplifies to $\partial B_5 \doteq 0$. In particular,

$$\partial b_{14}^5 = 0.$$

□

5.3. Cone points. In this subsection, we extend the definition of the cellular DGA to allow for Legendrians whose front projections have cone point singularities.

The standard cone $x_1^2 + x_2^2 = z^2$ is the (non-generic) front projection of a Legendrian cylinder in $J^1\mathbb{R}^2$. A point p on the front projection of a Legendrian $L \subset J^1S$ is called a **cone point** if there is a diffeomorphism of a neighborhood of p in $S \times \mathbb{R}$ onto a neighborhood of $(0,0,0) \in \mathbb{R}^3$ that takes the front projection of L to the standard cone. The inverse image in L of a cone point singularity is an S^1 , so that neighborhoods of cone points in L are topologically cylinders.

Consider a Legendrian $L \subset J^1S$ with generic base and front projections except for the presence of finitely many cone points whose base projections are disjoint from the image of cusps and crossings. Write $\Sigma_{\text{cone}} \subset S$ and $\Sigma \subset S$ for the base projection of cone points, and the remaining singular set of L . Let \mathcal{E} be a polygonal decomposition of $\pi_x(L) \subset S$ that contains Σ (resp. Σ_{cone}) in its 1-skeleton (resp. 0-skeleton).

We associate a DGA, (\mathcal{A}, ∂) , to L as in Section 3 using \mathcal{E} with the following modification at a cone point. Assume that near a zero cell, $e_\alpha^0 \in \Sigma_{\text{cone}}$, sheets of L are labeled S_1, \dots, S_n with the cone point connecting sheets S_k and S_{k+1} . Note that a Maslov potential μ on L must have $\mu(S_k) = \mu(S_{k+1}) + 1$; see the desingularization of the cone point in Figure 13.

- (1) The generators associated to e_α^0 are $a_{i,j}^\alpha$ with $1 \leq i < j \leq n$ and $(i,j) \neq (k,k+1)$ with

$$|a_{i,j}^\alpha| = \mu(S_i) - \mu(S_j) - 1.$$

The corresponding upper triangular matrix, A , (the $(k,k+1)$ -entry is 0) satisfies $\partial A = A^2$. This same matrix is used when e_α^0 occurs as an initial or terminal vertex of a bordering 1-cell or 2-cell.

- (2) Choose a 2-cell e_β^2 that borders e_α^0 , and label one of the occurrences of e_α^0 as a vertex along ∂e_β^2 with a U . When computing ∂C for e_β^2 , we add an additional edge to ∂e_β^2 at this vertex, and insert the matrix

$$(30) \quad U = I + \sum_{\ell < k} a_{\ell,k+1} E_{\ell,k} + \sum_{k+1 < \ell} a_{k,\ell} E_{k+1,\ell}$$

into the product of edges that appears in ∂C .

Proposition 5.4. *The DGA (\mathcal{A}, ∂) is equivalent to the cellular DGA of L .*

Proof. By a small Legendrian isotopy, L becomes front and base generic with the cone point replaced by 4 swallowtail points, connected by 4-cusp edges, and 2-crossing arcs, cf. [7, Section 3.1]. In the base projection, the image of the cusp edges bounds a square, X , whose vertices are swallowtail points. Two opposite corners of the square are upward swallowtails on the lower sheet of the cone point, and

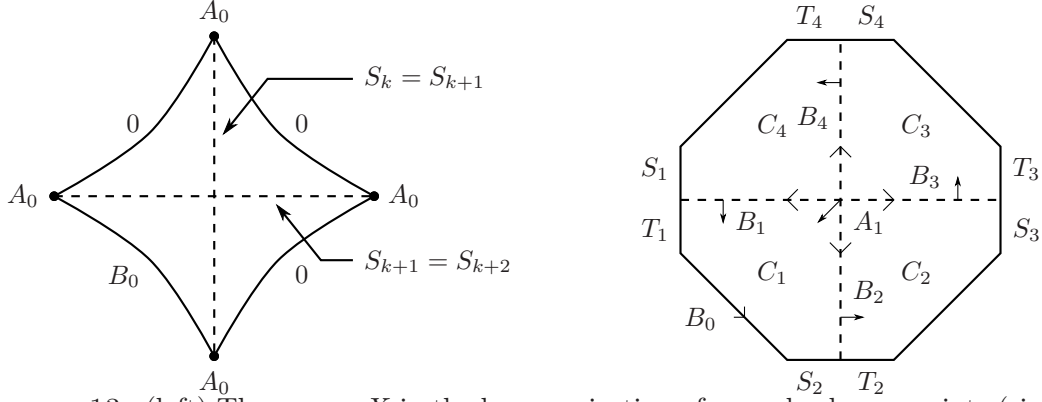


FIGURE 13. (left) The square X in the base projection of a resolved cone point. (right) Notation for matrices associated to cells of X and corners at swallowtail points.

the other two opposite corners are downward swallowtails on the upper sheet. Crossing arcs appear as diagonals of the square. See Figure 13. There are $n + 2$ sheets of L above the interior of X .

Produce from \mathcal{E} an L -compatible polygonal decomposition (for this generic perturbation of L) by replacing the single 0-cell, C , that was the cone point with the natural decomposition of the square X into 4 triangles from Figure 13. Modify the 1-cells that had previously had endpoints at C so that their endpoints are vertices of X in such a way that the corner labeled with U is replaced with some sequence of edges around the square starting with B_0 .

To prove the proposition we produce the cone point DGA from the cellular DGA via repeated application of Theorem 2.1. As a first step, we orient the boundary edges of X consistently, and then cancel the generators associated to 3 of the boundary edges of X with the generators of their terminal vertex. The generators and differentials of these edges are simply

$$\partial \boxed{B} = \boxed{A_+}(I + B) + (I + B)A_-$$

where all B and A matrices are $n \times n$ with diagonal entries given by the appropriate $b_{i,j}$ or $a_{i,j}$. (There are no 0 or 1 entries above the diagonal.) In the resulting quotient, generators of the three 1-cells have become 0, and generators associated to the 4 vertices of X are now equal. We use subscripts A_0 and B_0 for the matrices associated to the vertices of X and the 1 non-zero edge. See Figure 13.

Notations for cells in the interior of X are indicated in Figure 13. Above the 1-cells and 0-cells in the interior of X , we label sheets from 1 to $n + 2$ as they appear above the neighboring 2-cell indicated by the small arrows in Figure 13. This ordering specifies the notation for generators. *In this proof, the notation for matrices is fixed*, so that, eg., the matrix B_i always has rows and columns ordered according to the ordering of sheets above C_i , and A_1 is always ordered as above C_1 . The crossing locus results in the following 0 entries above the diagonal:

Matrix	0 entries
B_1, B_3	$(k + 2, k + 3)$
B_2, B_4	$(k, k + 1)$
A_1	$(k, k + 1)$ and $(k + 2, k + 3)$

As in Section 3.13, for computing differentials of 2-cells containing swallowpoints, we add additional edges at corners labeled with S or T . We notate the matrices associated to these edges as illustrated in Figure 13. These matrices are

$$S_1 = S_3 = I + E_{k+2,k+3} + \sum_{l < k+1} a_{l,k+1}^0 E_{l,k+1}, \quad T_1 = T_3 = I + E_{k+2,k+3}$$

$$S_2 = S_4 = I + E_{k,k+1} + \sum_{k+2 < l} a_{k,l-2}^0 E_{k+2,l}, \quad T_2 = T_4 = I + E_{k,k+1},$$

We denote the A_T matrices (as in (8) from 3.11) associated to the endpoints of the 1-cell B_i as

$$(A_0)_{T_i} = T_i(\widehat{A_0})_{(k+1,k+2)}T_i$$

For the 2-cells C_1 (resp. C_i , $2 \leq i \leq 4$), we choose initial and terminal vertices to be the endpoints of B_0 (resp. of B_i). We have differentials

$$\begin{aligned}\partial B_1 &= (A_0)_{T_1} (I + B_1) + (I + B_1) A_1, \\ \partial C_1 &= (\widehat{A_0})_{k+1, k+2} C_1 + C_1 (\widehat{A_0})_{k+1, k+2} + (I + \widehat{B_0})_{k+1, k+2} + S_2 Q_{(01)} (I + B_2) Q_{(01)} (I + B_1)^{-1} T_1, \\ \partial C_2 &= (A_0)_{T_2} C_2 + C_2 Q_{(01)} A_1 Q_{(01)} + (I + B_2) + T_2 S_3 Q_{(23)} (I + B_3) Q_{(23)}, \\ \partial C_3 &= (A_0)_{T_3} C_3 + C_3 Q_{(01)(23)} A_1 Q_{(01)(23)} + (I + B_3) + T_3 S_4 Q_{(01)} (I + B_4) Q_{(01)}, \\ \partial C_4 &= (A_0)_{T_4} C_4 + C_4 Q_{(23)} A_1 Q_{(23)} + (I + B_4) + T_4 S_1 Q_{(23)} (I + B_1) Q_{(23)},\end{aligned}$$

where the matrices $Q_{(01)}$ and $Q_{(23)}$ are permutation matrices for the transpositions $(k \ k+1)$ and $(k+2 \ k+3)$, and $Q_{(01)(23)} = Q_{(01)} Q_{(23)}$.

We begin to cancel generators. First, observe that

$$\partial \boxed{c_{k, k+1}^4} = 1 + a_{k, k+1}^0 + \boxed{b_{k, k+1}^1},$$

and use Theorem 2.1 to cancel the boxed generators. [The product of strictly upper triangular matrices has all $(i, i+1)$ -entries 0, so the $(k, k+1)$ -entry of ∂C_4 is the sum of the $(k, k+1)$ -entries from $I + B_4$, T_4 , S_1 , and $Q_{(23)}(I + B_1)Q_{(23)}$.] The remaining non-zero entries of C_4 are in correspondence with those of B_4 , so we cancel

$$\partial \boxed{C_4} = \boxed{B_4} + \dots$$

[The condition on the ordering of generators required to apply Theorem 2.1 is verified as in proof of Theorem 4.3.] Note that in the quotient, we have

$$b_{k+2, k+3}^4 \doteq 1,$$

so we can compute and cancel

$$\partial \boxed{c_{k+2, k+3}^3} = 1 + a_{k, k+1}^0 + b_{k+2, k+3}^4 \doteq \boxed{a_{k, k+1}^0}.$$

Record that

$$a_{k, k+1}^0 \doteq 0, \Rightarrow b_{k, k+1}^1 \doteq 1.$$

With non-zero entries of C_3 and B_3 now in correspondence, cancel

$$\partial \boxed{C_3} = \boxed{B_3} + \dots$$

This gives $b_{k, k+1}^3 \doteq 1$, so that

$$\partial c_{k, k+1}^2 = 1 + a_{k, k+1}^0 + b_{k, k+1}^3 \doteq 0.$$

Hence, we can modify our usual orderings of generators (arising as in Lemma 4.1) by moving $c_{k, k+1}^2$ to be the first generator. This allows us to cancel all entries of C_2 *other than* $c_{k, k+1}^2$ (the $(k, k+1)$ entry of B_2 is 0) by

$$\partial \boxed{C_2} = \boxed{B_2} + \dots$$

With $b_{k, k+1}^1$ already canceled, the remaining generators in B_1 are in correspondence with those from A_1 ; cancel

$$\partial \boxed{B_1} = \boxed{A_1} + \dots$$

To summarize where we are, the only remaining generators associated to the cells of X are $c_{k, k+1}^2$ and the entries of B_1 , C_1 , and A_0 except for $a_{k, k+1}^0$ which has $a_{k, k+1}^0 \doteq 0$. We have

$$C_4 \doteq 0, \quad C_3 \doteq 0, \quad C_2 \doteq c_{k, k+1}^2 E_{k, k+1}, \quad (I + B_1) \doteq (I + E_{k, k+1}).$$

Using $\partial C_4 \doteq 0$ gives

$$I + B_4 \doteq T_4 S_1 Q_{(23)} (I + E_{k, k+1}) Q_{(23)} = S_1.$$

Then, $\partial C_3 \doteq 0$ gives

$$I + B_3 \doteq T_3 S_4 Q_{(01)} (I + B_4) Q_{(01)} \doteq T_3 S_4 Q_{(01)} S_1 Q_{(01)} \doteq$$

$$(I + E_{k+2,k+3})(I + E_{k,k+1} + \sum_{k+3 < l} a_{k,l-2}^0 E_{k+2,l})(I + E_{k+2,k+3} + \sum_{l < k} a_{l,k+1}^0 E_{l,k}) =$$

$$I + E_{k,k+1} + \sum_{k+3 < l} a_{k,l-2}^0 E_{k+2,l} + \sum_{l < k} a_{l,k+1}^0 E_{l,k}.$$

[The range of l in the summations is different from in the definitions of S_1 and S_4 because $a_{k,k+1}^0 \doteq 0$.]
Next, compute

$$I + B_2 \doteq F + G$$

where

$$F = T_2 S_3 Q_{(23)}(I + B_3) Q_{(23)}$$

$$\doteq (I + E_{k,k+1})(I + E_{k+2,k+3} + \sum_{l < k} a_{l,k+1}^0 E_{l,k+1})(I + E_{k,k+1} + \sum_{k+3 < l} a_{k,l-2}^0 E_{k+3,l} + \sum_{l < k} a_{l,k+1}^0 E_{l,k})$$

$$= I + E_{k+2,k+3} + \sum_{l < k} a_{l,k+1}^0 (E_{l,k} + E_{l,k+1}) + \sum_{k+3 < l} a_{k,l-2}^0 (E_{k+2,l} + E_{k+3,l}),$$

and

$$G = (A_0)_{T_2} C_2 + C_2 Q_{(01)} A_1 Q_{(01)}.$$

Using that $\partial B_1 = 0$ and $B_1 \doteq (I + E_{k,k+1})$ shows that

$$A_1 \doteq (I + B_1)^{-1} (A_0)_{T_1} (I + B_1) \doteq (I + E_{k,k+1} + E_{k+2,k+3}) \widehat{A_0}_{(k+1,k+2)} (I + E_{k,k+1} + E_{k+2,k+3}),$$

so we compute

$$C_2 Q_{(01)} A_1 Q_{(01)} \doteq (c_{k,k+1}^2 E_{k,k}) A_1 Q_{(01)}$$

$$\doteq \left(\sum_{k+1 < l} c_{k,k+1}^2 a_{k,l}^0 E_{k,l+2} \right) + c_{k,k+1}^2 E_{k,k+2} + c_{k,k+1}^2 E_{k,k+3}; \text{ and}$$

$$(A_0)_{T_2} C_2 \doteq \sum_{l < k} a_{l,k}^0 c_{k,k+1}^2 E_{l,k+1}.$$

Combining the previous computations gives

$$I + B_2 \doteq I + E_{k+2,k+3} + \sum_{l < k} a_{l,k+1}^0 (E_{l,k} + E_{l,k+1}) + \sum_{k+3 < l} a_{k,l-2}^0 (E_{k+2,l} + E_{k+3,l}) +$$

$$c_{k,k+1}^2 E_{k,k+2} + c_{k,k+1}^2 E_{k,k+3} + \sum_{k+1 < l} c_{k,k+1}^2 a_{k,l}^0 E_{k,l+2} + \sum_{l < k} a_{l,k}^0 c_{k,k+1}^2 E_{l,k+1}.$$

Observe that $Q_{(01)}(I + B_2)Q_{(01)}$ is upper triangular with $(k+1, k+2)$ -entry $c_{k,k+1}^2$.

Next, since the $(k+1, k+2)$ -entries of S_2 , $(I + B_1)^{-1}$, and T are all 0, the formula for ∂C_1 shows

$$\partial \boxed{c_{k+1,k+2}^1} = \boxed{c_{k,k+1}^2}.$$

After making this cancellation,

$$Q_{(01)}(I + B_2)Q_{(01)} \doteq I + E_{k+2,k+3} + \sum_{l < k} a_{l,k+1}^0 (E_{l,k} + E_{l,k+1}) + \sum_{k+3 < l} a_{k,l-2}^0 (E_{k+2,l} + E_{k+3,l}),$$

and another careful matrix multiplication shows

$$S_2 Q_{(01)}(I + B_2) Q_{(01)}(I + B_1)^{-1} T_1 = I + \sum_{l < k} a_{l,k+1}^0 E_{l,k} + \sum_{k+3 < l} a_{k,l-2}^0 E_{k+3,l}.$$

This matrix has all above diagonal entries in the $k+1$ and $k+2$ rows and columns equal to 0. Thus, the entries of ∂C^1 in these rows and columns agree with the entries of $(\widehat{A_0})_{k+1,k+2} C_1 + C_1 (\widehat{A_0})_{k+1,k+2}$, and we have, for $i < k+1$ and $k+2 < j$,

$$\partial \boxed{c_{i,k+2}^1} \doteq \sum_{i < m < k+1} a_{i,m}^0 c_{m,k+2}^1 + \boxed{c_{i,k+1}^1}; \quad \partial \boxed{c_{k+1,j}^1} \doteq \boxed{c_{k+2,j}^1} + \sum_{k+2 < m < j} c_{k+1,m}^1 a_{m-2,j-2}^0.$$

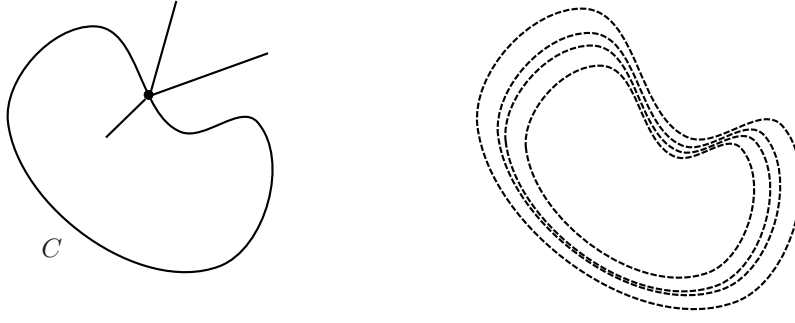


FIGURE 14. (left) The curve C in the decomposition \mathcal{E}_C . (right) Crossing arcs appearing in the neighborhood $N \cong S^1 \times [0, 1]$ of C .

Here, we cancel beginning with $i = k$ (resp. $j = k + 3$), and then with i decreasing (resp. j increasing), so that inductively when we cancel the equation simplifies to

$$\partial \boxed{c_{i,k+2}^1} \doteq \boxed{c_{i,k+1}^1}; \quad \partial \boxed{c_{k+1,j}^1} \doteq \boxed{c_{k+2,j}^1}.$$

Thus, all of the $c_{i,k+1}^1$, $c_{i,k+2}^1$, $c_{k+1,j}^1$, $c_{k+2,j}^1$, and $c_{k+1,k+2}^1$ are 0 in the quotient. The remaining generators from C^1 are in correspondence with the B_0 generators, and we cancel

$$\partial \boxed{C_1} = \boxed{\widehat{B_{0(k+1,k+2)}}} + \cdots.$$

In the quotient, we have $C_1 \doteq 0$ and

$$I + \widehat{B_{0(k+1,k+2)}} = I + \sum_{l < k} a_{l,k+1}^0 E_{l,k} + \sum_{k+3 < l} a_{k,l-2} E_{k+3,l},$$

which gives the relations

$$I + B_0 \doteq I + \sum_{l < k} a_{l,k+1}^0 E_{l,k} + \sum_{k+1 < l} a_{k,l}^0 E_{k+1,l} = U.$$

At this point all generators above the square X have been canceled except for $a_{i,j}^0$ with $(i, j) \neq (k, k + 1)$. Moreover, the matrix associated to each of the 4 corners of X is A_0 , and to all of the outer edges is 0 except for the edge labeled B_0 in Figure 13 which has $I + B_0 = U$ with U as in the definition of the DGA with cone points. Thus, the appearances of $a_{i,j}^0$ in the differential of bordering cells matches the definition of the cone point DGA. \square

5.4. 1-cells with more than one crossing or cusp arc. Often it is possible to relax the requirement that at most 1 crossing or cusp arc lies above each 1-cell. We do not attempt to give the most general statement here, but restrict ourselves to an illustrative example that will be useful for the class of examples in Section 5.6.

For $L \subset J^1 S$, suppose that \mathcal{E}_C is a polygonal decomposition of $\pi_x(L)$ that is L -compatible except near a 2-sided simple closed curve C contained in the 1-skeleton of \mathcal{E}_C . In a neighborhood $N \cong S^1 \times [0, 1]$ of $C \cong S^1 \times \{1/2\}$, there appear several crossing arcs that project to small shifts of C in the normal direction, and no other crossing or cusp arcs. See Figure 14. Above N , L is a union of sheets that project homeomorphically to N . Suppose further that *no two sheets of L cross each other more than once above N* .

With this set-up, we associate a variant of the cellular DGA $(\mathcal{A}_C, \partial)$ to \mathcal{E}_C . For definiteness of notation, choose one side of C , and label the sheets of L as S_1, \dots, S_n in the order they appear above this side of N . Let $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ be the permutation such that S_i appears in position $S_{\sigma(i)}$ on the other side of N . Note that for $i < j$, S_i and S_j cross above S if and only if $\sigma(i) > \sigma(j)$. To each 1-cell (resp. 0-cell) of C , assign generators $b_{i,j}$ (resp. $a_{i,j}$) for all $i < j$ with $\sigma(i) < \sigma(j)$. (So multiple pairs of crossing sheets contribute to multiple 0s to the upper triangle, as in the single crossing case.) The differential is defined as in the case of the cellular DGA: 0-cells in C have $\partial A = A^2$; 1-cells in C have $\partial B = A_+(I + B) + (I + B)A_-$; when a 1-cell or 0-cell appear in the boundary of a 2-cell, we place the generators into matrices B_i or A_{v_i} being sure to order rows and columns using the



FIGURE 15. (Far left) A cross-section of the front projection of L above $\{0\} \times [0, 1] \subset N$, with $\sigma_0 = (1\ 2)$, $\sigma_1 = (2\ 3)$, $\sigma_2 = (3\ 4)$, $\sigma_3 = (2\ 3)$, and $\sigma = (1\ 4\ 2)$. (Middle) The decomposition of C in \mathcal{E}_C . (Right) The decomposition \mathcal{E}_L in the neighborhood N .

ordering of sheets above the 2-cell. For instance, above 2-cells on the side of C where sheets of L above N appear in order S_1, \dots, S_n (resp. $S_{\sigma^{-1}(1)}, \dots, S_{\sigma^{-1}(n)}$) generators associated to a bordering 1-cell in C are placed as $B_i = (b_{i,j})$ (resp. $B_i = Q_{\sigma}^{-1}(b_{i,j})Q_{\sigma} = (b_{\sigma^{-1}(i), \sigma^{-1}(j)})$ where Q_{σ} is the permutation matrix $Q_{\sigma} = \sum E_{\sigma(i), i \cdot}$)

Proposition 5.5. *The DGA (\mathcal{A}, ∂) is stable tame isomorphic to the cellular DGA of L .*

Proof. For simplicity, we consider the case when the decomposition of C consists of a single 0-cell at $0 = 1 \in S^1 = \mathbb{R}/\mathbb{Z}$ and 1-cell at $(0, 1)$ in detail, with a similar argument applying in general. We produce a L -compatible polygonal decomposition \mathcal{E}_L from \mathcal{E}_C by replacing the decomposition of the single closed curve, C , with a decomposition as in Figure 15; using coordinates $(x, y) \in S^1 \times [0, 1] \cong N$, we take the crossing arcs in L to be at $y = y_0, y = y_1, \dots, y = y_r$, each decomposed into a 0-cell, $\{(0, y_i)\}$ and a 1-cell $(0, 1) \times \{y_i\}$. Additional 1-cells are given by $\{0\} \times (y_{m-1}, y_m)$ running perpendicularly through the annulus connecting the 0-cells of the different crossing arcs. See Figure 15

Denote the Cellular DGA of \mathcal{E}_L by $(\mathcal{A}_L, \partial)$. We place generators of \mathcal{A}_L corresponding to the translates of the original 0- and 1-cell of C into matrices A_m, B_m , for $m = 0, \dots, r$, and place the generators corresponding to vertical product cells of the form $\{0\} \times (y_{m-1}, y_m)$ (resp. $(0, 1) \times (y_{m-1}, y_m)$) into matrices B'_m, C_m , for $m = 1, \dots, r$. For the A_m and B_m we take the ordering of sheets (and labeling of generators) to agree with the ordering in the bordering region that has smaller y -coordinate. *In this proof, these notations for matrices are fixed* (including the ordering of rows and columns).

To describe the differential, let $\sigma_0 = (k_0, k_0 + 1), \dots, \sigma_r = (k_r, k_r + 1)$ be the sequence of transpositions corresponding to the crossing arcs as y increases from 0 to 1. So, if the crossings occur at $y = y_0, \dots, y = y_r$, then the sheet in position i immediately before y_m is in position $\sigma_m(i)$ after y_m . Notice that the products

$$\tilde{\sigma}_m = \sigma_m \sigma_{m-1} \cdots \sigma_0$$

then have the property that the sheet labeled S_i before y_0 is labeled $S_{\tilde{\sigma}_m(i)}$ after y_m . In particular, the permutation σ defined prior to the statement of the proposition is $\sigma = \tilde{\sigma}_r$.

For the subalgebra generated by the A_m and B'_m generators (above $\{0\} \times [0, 1]$) we have

$$\partial B'_m = A_m B'_m + B'_m Q_{\sigma_{m-1}} A_{m-1} Q_{\sigma_{m-1}}^{-1}.$$

[The (i, j) -entry of $Q_{\sigma_{m-1}} A_{m-1} Q_{\sigma_{m-1}}^{-1}$ is $a_{\sigma_{m-1}^{-1}(i), \sigma_{m-1}^{-1}(j)}^{m-1}$.] We cancel all of the B'_m and A_m generators with $1 \leq m \leq r$ and some of the A_0 generators via the following inductive procedure.

To begin, for $(i, j) \neq (k_1, k_1 + 1)$, cancel $b_{i,j}^1$ with $a_{i,j}^1$, and for $(i, j) = (k_1, k_1 + 1)$, cancel $b_{k_1, k_1+1}^1 = a_{\sigma_0^{-1}(k_1), \sigma_0^{-1}(k_1+1)}^0$ [We apply Theorem 2.1 repeatedly, so that $|j - i|$ increases as we go. Thus, any $b_{p,q}$ terms that appear in $\partial b_{i,j}$ are already 0 when the theorem is applied.]

For $m \geq 2$, inductively suppose that we have canceled all of the generators of B'_l with $l < m$, and A_l with $0 < l < m$, as well as all entries $a_{i,j}^0$ with the property that the sheets labeled S_i and S_j before y_0 cross one another at some $y \in [y_0, y_m)$. Moreover, suppose that in the quotient, we have

$$A_{m-1} \doteq Q_{\tilde{\sigma}_{m-2}} A_0 Q_{\tilde{\sigma}_{m-2}}^{-1},$$

and $a_{i,j}^0 \doteq 0$ when S_i and S_j cross before y_m .

Then, the differential becomes

$$\partial B'_m = A_m(I + B'_m) + (I + B'_m)Q_{\sigma_{m-1}}Q_{\tilde{\sigma}_{m-2}}A_0Q_{\tilde{\sigma}_{m-2}}^{-1}Q_{\sigma_{m-1}}^{-1}.$$

Note that the final term is $Q_{\tilde{\sigma}_{m-1}} A_0 Q_{\tilde{\sigma}_{m-1}}^{-1}$; it is upper-triangular (because of the entries that are equal to 0.) For $(i, j) \neq (k_m, k_m + 1)$, we cancel $b_{i,j}^m$ and $a_{i,j}^m$. For $(i, j) = (k_m, k_m + 1)$, we cancel

$$\partial b_{k_m, k_m+1}^m = a_{\tilde{\sigma}_{m-1}^{-1}(k_m), \tilde{\sigma}_{m-1}^{-1}(k_m+1)}^0.$$

[This entry has not yet been canceled because of the assumption that no two sheets cross each other more than once in N .] In the quotient, since $B'_m \doteq 0$, it follows that

$$0 \doteq \partial B'_m = A_m + Q_{\tilde{\sigma}_{m-1}} A_0 Q_{\tilde{\sigma}_{m-1}}^{-1} \Rightarrow A_m = Q_{\tilde{\sigma}_{m-1}} A_0 Q_{\tilde{\sigma}_{m-1}}^{-1},$$

and $a_{\tilde{\sigma}_{m-1}^{-1}(k_m), \tilde{\sigma}_{m-1}^{-1}(k_m+1)}^0 \doteq 0$ so that the induction proceeds.

At the conclusion of the procedure, the only remaining generators are those entries of A_0 corresponding to sheets that do not cross one another above N , and we have

$$(31) \quad A_r \doteq Q_\sigma A_0 Q_\sigma^{-1}.$$

At this point, differentials of the C_m have simplified as

$$\begin{aligned} \partial C_m &= A_m C_m + C_m Q_{\sigma_{m-1}} A_{m-1} Q_{\sigma_{m-1}}^{-1} + (I + B_m)(I + B'_m) + (I + B'_m)(I + Q_{\sigma_{m-1}} B_{m-1} Q_{\sigma_{m-1}}^{-1}) \\ &\doteq A_m C_m + C_m Q_{\sigma_{m-1}} A_{m-1} Q_{\sigma_{m-1}}^{-1} + B_m + Q_{\sigma_{m-1}} B_{m-1} Q_{\sigma_{m-1}}^{-1}. \end{aligned}$$

Then, using a similar inductive procedure, we can cancel all of the C_m and B_m for $1 \leq m \leq r$, as well as those entries of B_0 corresponding to sheets S_i and S_j that cross somewhere in N . In the quotient, we have the relation

$$(32) \quad B_r \doteq Q_\sigma B_0 Q_\sigma^{-1}.$$

The remaining generators in this final quotient of $(\mathcal{A}_L, \partial)$ are exactly as in the description of $(\mathcal{A}_C, \partial)$, and the differentials agree as well in view of the relations (31) and (32). \square

5.4.1. Extension to multiple cone points. The DGA with cone points from Proposition 5.4 has a similar easy extension to the case where L has several cone points projecting to the same point $x_0 \in S$. Suppose that above x_0 there are r cone points joining the pairs of sheets $(k_1, k_1 + 1), \dots, (k_r, k_r + 1)$. In this case, form a DGA as in Proposition 5.4 with the two modifications:

- (1) The 0-cell at x_0 has generators $a_{i,j}$ with $i < j$ and $(i, j) \neq (k_p, k_p + 1)$ for all $1 \leq p \leq r$.
- (2) The matrix $U = U_r U_{r-1} \cdots U_1$ where

$$(33) \quad U_p = I + \sum_{\ell < k_p} a_{\ell, k_p+1} E_{\ell, k_p} + \sum_{k_p+1 < \ell} a_{k_p, \ell} E_{k_p+1, \ell}.$$

Proposition 5.6. *The resulting DGA is equivalent to the DGA from Proposition 5.4.*

Proof. To see this, isotope the front projection of L so that the cone points appear at distinct points of S in a small neighborhood of x_0 . Then, form a cellular decomposition where the single 0-cell at x_0 is expanded to a string of 0-cells, A_1, \dots, A_r at cone points connected by a sequence of 1-cells, B_1, \dots, B_{r-1} , all located in the corner of x_0 that was labeled with U . All of the generators for the B_1, \dots, B_{r-1} , and A_2, \dots, A_r , as well as the $(k_p, k_p + 1)$ -entries of A_1 are canceled by a sequence of applications of Theorem 2.1. (At the inductive step, cancel B_m with all of the generators of A_{m+1} and the $(k_l, k_l + 1)$ entries of A_m for $l \geq m + 1$.) When computing the differential of 2-cell that contains the string of 0-cells and 1-cells, the contribution to the product of edges from this corner is the product of the individual U_p matrices

$$U_p = I + \sum_{\ell < k_p} a_{\ell, k_p+1} E_{\ell, k_p} + \sum_{k_p+1 < \ell} a_{k_p, \ell} E_{k_p+1, \ell},$$

and in the quotient $a_{i,j}^p = a_{i,j}^q$ for all $1 \leq p < q \leq r$. \square

5.5. Front spinning a Legendrian knot. In this subsection, we show how the DGA of a surface which is the front-spin of a 1-dimensional Legendrian knot can be derived from the DGA of the knot. The result agrees with known computations when the axis of rotation and the knot do not intersect, and provides new examples when the axis and knot do intersect.

5.5.1. *The suspension of a DGA.* We begin with some algebraic preliminaries, which are motivated by a homotopy theory for DGAs (or more generally for differential graded operad algebras where the operad might not be the associative one). Suppose $(\mathcal{A}, \partial_{\mathcal{A}})$ is a DGA freely generated by a_1, \dots, a_k , triangular with respect to these generators (see Section 2.1.1), possibly unital, and with an arbitrary coefficient ring. Define the **suspension** of $(\mathcal{A}, \partial_{\mathcal{A}})$, $(\hat{\mathcal{A}}, \partial_{\hat{\mathcal{A}}})$, as follows:

- $\hat{\mathcal{A}}$ is freely generated by a_i, \hat{a}_i ;
- the grading $|a_i|$ of $a_i \in \mathcal{A}$ is the same as it is in \mathcal{A} , while $|\hat{a}_i| = |a_i| + 1$;
- $\partial_{\hat{\mathcal{A}}}(a_i) = \partial_{\mathcal{A}}(a_i)$ and $\partial_{\hat{\mathcal{A}}}(\hat{a}_i) = \Gamma(\partial_{\mathcal{A}}(a_i))$ where $\Gamma : \mathcal{A} \rightarrow \hat{\mathcal{A}}$ is a derivation defined by its image of the generators $\Gamma(a_i) = \hat{a}_i$. (So for example, $\Gamma(1) = 0$ if \mathcal{A} is unital, and $\Gamma(a_1 a_2) = \hat{a}_1 a_2 + (-1)^{|a_1|} a_1 \hat{a}_2$.)

We assume $\hat{\mathcal{A}}$ is unital if \mathcal{A} is.

We also consider a variation of this construction. Let $(\mathcal{L}, \partial_{\mathcal{L}})$ be a sub-DGA of $(\mathcal{A}, \partial_{\mathcal{A}})$ generated by some subset of the generating set. Define the **suspension of \mathcal{A} relative to \mathcal{L}** to be the DGA $\hat{\mathcal{A}}$ as above except that for all elements $l \in \mathcal{L}$, $\hat{l} = 0$.

Lemma 5.1. *If $(\mathcal{A}, \partial_{\mathcal{A}})$ and $(\mathcal{B}, \partial_{\mathcal{B}})$ are stable tame isomorphic then so too are $(\hat{\mathcal{A}}, \partial_{\hat{\mathcal{A}}})$ and $(\hat{\mathcal{B}}, \partial_{\hat{\mathcal{B}}})$. If $(\mathcal{L}, \partial_{\mathcal{L}})$ is a sub-DGA as above of both $(\mathcal{A}, \partial_{\mathcal{A}})$ and $(\mathcal{B}, \partial_{\mathcal{B}})$ and the elementary automorphisms in the definition of stable-tame isomorphism fixes \mathcal{L} , then the suspensions of \mathcal{A} and \mathcal{B} relative to \mathcal{L} are stable tame isomorphic.*

Proof. It is easy to check that if $\mathcal{B} = S(\mathcal{A})$ is a stabilization with new generators x, y then $\hat{\mathcal{B}} = S(S(\hat{\mathcal{A}}))$ with new generators $x, y, \hat{x} = \Gamma(x), \hat{y} = \Gamma(y)$. So assume $\phi : (\mathcal{A}, \partial_{\mathcal{A}}) \rightarrow (\mathcal{B}, \partial_{\mathcal{B}})$ is an elementary isomorphism, defined on generators as $\phi(a_i) = a_i + \delta_j^i w$. (In the language of Section 2.1.1, $\phi(a_j) = a_j + w$ while ϕ is the identity on all other generators.) We claim the algebra isomorphism $\hat{\phi} : \hat{\mathcal{A}} \rightarrow \hat{\mathcal{B}}$ defined on generators by

$$\hat{\phi}(a_i) = a_i + \delta_j^i w, \quad \hat{\phi}(\Gamma(a_i)) = \Gamma(a_i) + \delta_j^i \Gamma(w)$$

is a (tame) DGA isomorphism.

First note that $\Gamma\phi = \hat{\phi}\Gamma$. This is obvious on generators, while the inductive (on word-length) step follows from linearity and the following relation on monomials:

$$\begin{aligned} \hat{\phi}(\Gamma(xy)) &= \hat{\phi}(\Gamma(x)y + (-1)^{|x|}x\Gamma(y)) = \Gamma(\phi(x))\phi(y) + (-1)^{|x|}\phi(x)\Gamma(\phi(y)) \\ &= \Gamma(\phi(x))\phi(y) + (-1)^{|\phi(x)|}\phi(x)\Gamma(\phi(y)) = \Gamma(\phi(xy)). \end{aligned}$$

First note that since $\partial_{\mathcal{B}} = \phi\partial_{\mathcal{A}}\phi^{-1}$ then for generator $a_i \in \hat{\mathcal{A}}$, $\hat{\phi}\partial_{\hat{\mathcal{A}}}\hat{\phi}^{-1}(a_i) = \partial_{\hat{\mathcal{B}}}(a_i)$. For the other generators,

$$\begin{aligned} \hat{\phi}\partial_{\hat{\mathcal{A}}}\hat{\phi}^{-1}(\Gamma(a_i)) &= \hat{\phi}\partial_{\hat{\mathcal{A}}}(\Gamma(a_i) - \delta_j^i \Gamma(w)) \\ &= \hat{\phi}\Gamma(\partial_{\mathcal{A}}(a_i)) - \delta_j^i \hat{\phi}\Gamma(\partial_{\mathcal{A}}(w)) \\ \partial_{\hat{\mathcal{B}}}(\Gamma(a_i)) &= \Gamma(\partial_{\mathcal{B}}(a_i)) \\ &= \Gamma(\phi\partial_{\mathcal{A}}\phi^{-1}(a_i)) \\ &= \Gamma(\phi\partial_{\mathcal{A}}(a_i - \delta_j^i w)) \\ &= \Gamma\phi(\partial_{\mathcal{A}}(a_i)) - \delta_j^i \Gamma\phi(\partial_{\mathcal{A}}(w)) \end{aligned}$$

These are equal by the $\Gamma\phi = \hat{\phi}\Gamma$ relation above.

If the suspensions are relative, the same formulas and arguments apply after recalling that $\Gamma(l) = 0$ for $l \in \mathcal{L}$. \square

5.5.2. *Spun cellular decomposition.* Let $J^1(\mathbb{R}_{x_1}) = (\mathbb{R}_{x_1, y_1, z}^3, \ker\{dz - y_1 dx_1\})$ denote the standard contact structure. Let $\pi_{x_1, z} : J^1(\mathbb{R}_{x_1}) \rightarrow J^0(\mathbb{R}_{x_1}) = \mathbb{R}_{x_1, z}^2$ be the lower-dimensional analog of the front projection $\pi_{x, z}$. Let $\Lambda \subset J^1(\mathbb{R}_{x_1})$ be a Legendrian embedding of the union of (a non-negative) number of circles sitting in $\{x_1 < 0\}$ as well as a (non-negative) number of arcs in $\{x_1 \leq 0\}$ whose endpoints sit on the z -axis. We impose the following condition on the front projections of the arcs in $J^0(\mathbb{R}_{x_1})$. If we reflect the arcs' fronts through the z -axis, the arcs and their reflections must glue together to form the front of a collection of (smooth) Legendrian circles. An arbitrary Legendrian

embedding of circles and arcs can achieve these conditions with a Legendrian isotopy. Embed $J^0(\mathbb{R}_{x_1})$ as $\{x_2 = 0\} \subset J^0(\mathbb{R}_{x_1, x_2}^2) = \mathbb{R}_{x_1, x_2, z}^3$. Consider the rotation about the z -axis, $S^1 \times \pi_{x, z}(\Lambda) \subset J^0(\mathbb{R}_{x_1, x_2}^2)$ as defined in [15, Section 4]. We say that the **front spin** of Λ is the Legendrian surface L in standard contact $J^1(\mathbb{R}_{x_1, x_2}^2) = \mathbb{R}^5$ whose front is given by this rotation. Note that L is a Legendrian embedding of a collection of spheres and/or tori. In the case when two arc endpoints have the same front projection in $\{x_1 = 0\} \subset J^0(\mathbb{R}_{x_1})$, then the spun surface L has a cone point. We consider Legendrians with possibly multiple pairs of matching arc-front-endpoint conditions, but we assume no three arcs have fronts with the same endpoint.

Consider a cellular decomposition of $\pi_{x_1}(\Lambda) \subset \mathbb{R}_{x_1}^1$, where in the case that Λ has a Legendrian arc, the decomposition includes $\{x_1 = 0\}$ as a 0-cell. For each cell e of $\pi_x(\Lambda)$, excluding the 0-cell $\{x_1 = 0\}$ if it appears, we associate two cells for $\pi_x(L) \subset \mathbb{R}_{x_1, x_2}^2$, which we denote by e and \hat{e} . The first is simply the cell e after embedding $J^0(\mathbb{R}_{x_1})$ as $\{x_2 = 0\} \subset J^0(\mathbb{R}_{x_1, x_2}^2)$. The second is the embedded e spun about the z -axis; that is, a radially symmetric map of $e \times [-\pi, \pi]$ with the ends $e \times \{-\pi\}$ and $e \times \{\pi\}$ mapping to the same cell as e . Note that the crossing/cusp 0-cells become analogous 1-cells under spinning.

If $e = \{x_1 = 0\}$ is a 0-cell of Λ , then we denote by e again the 0-cell $\{x_1 = 0 = x_2\}$ of L . As mentioned above, if $e = \{x_1 = 0\}$ represents a crossing 0-cell of Λ , then the corresponding cell in L sits below a cone point.

5.5.3. The cellular DGA of a front spin.

Proposition 5.1. *The LCH DGA $(\mathcal{A}_{LCH, \Lambda}, \partial_{LCH, \Lambda})$ of the 1-dimensional Legendrian Λ is stable-tame isomorphic to the cellular DGA of Λ , $(\mathcal{A}_\Lambda, \partial_\Lambda)$, as specified in Section 3 equations (2) and (3).*

Proof. This can be deduced from a (greatly shortened) argument along the lines of the isomorphism between the Cellular DGA and LCH proved for 2-dimensional Legendrians in [35].

However, this is not necessary, since in the 1-dimensional case the Cellular DGA is almost identical to the LCH DGA as computed after the addition of *splashes* or *dips*, [23], [37]. To give a definite isomorphism, we use the version of splashes from [24] where the generators and differentials for the LCH DGA of a 1-dim Legendrian $\Lambda \subset J^1\mathbb{R}$ are stated in Proposition 4.1 of [24]. There Λ is broken into a product of elementary tangles containing either a single left cusp, right cusp, or crossing. To the right of the m -th elementary tangle, a collection of splashes is given (for all m), and this produces two upper triangular matrices of generators labeled in [24] as $x_{m; i, j}^+$ and $x_{m; i, j}^-$. In addition, when there is a crossing (resp. right cusp) in the m -th elementary tangle, there is a single generator y_m (resp. z_m).

A Λ -compatible polygonal decomposition of $\pi_x(\Lambda)$ arises in an obvious way by taking the 0-cells to occur at cusps and crossings. Label the 0- and 1-cells, so that from left to right they appear as $A_1, B_2, A_2, B_3, A_3, \dots, B_{N+1}, A_{N+1}$. If we identify the Cellular DGA generators $a_{i, j}^m$ and $b_{i, j}^m$ with $x_{m; i, j}^+$ and $x_{m; i, j}^-$ respectively, then the differentials almost agree. The differences are the following:

- (1) At 0-cells with crossings, there are 2 more generators, y_m and $x_{m-1; k, k+1}^+$, in the LCH DGA.
- (2) At 0-cells with right cusps, there are more generators, z_m , $x_{m-1; k, k+1}^+$, $x_{m-1; i, k+1}^+$, $x_{m-1; i, k}^+$, $x_{m-1; k, j}^+$, and $x_{m-1; k+1, j}^+$.

Repeated applications of Theorem 2.1 (that the reader may by now find routine) cancel these generators in pairs. In this quotient of the LCH DGA, the differential agrees precisely with the Cellular DGA differential. □

We can now state the main proposition of Subsection 5.5

Proposition 5.2. (1) *Suppose Λ has no arc components. The DGA $(\mathcal{A}_L, \partial_{\mathcal{A}_L})$ is the suspension of $(\mathcal{A}_\Lambda, \partial_{\mathcal{A}_\Lambda})$.*
 (2) *Suppose Λ has arc components whose front projections have distinct endpoints. The DGA $(\mathcal{A}_L, \partial_{\mathcal{A}_L})$ is the suspension of $(\mathcal{A}_\Lambda, \partial_{\mathcal{A}_\Lambda})$ relative to the sub-DGA associated to the $\{x_1 = 0 = x_2\}$ 0-cell.*
 (3) *Suppose Λ has arc components whose front projections have r pairs of matching endpoints. The DGA $(\mathcal{A}_L, \partial_{\mathcal{A}_L})$ is the suspension of $(\mathcal{A}_\Lambda, \partial_{\mathcal{A}_\Lambda})$ relative to the sub-DGA associated to the*

$\{x_1 = 0 = x_2\}$ 0-cell, with the following modification. Let A be the matrix of generators above the $\{x_1 = 0 = x_2\}$ 0-cell. We replace $\Gamma(A) = 0$ with $\Gamma(A) = U - I$ from equation (33).

Proof. As we see from Sections 3.6.1 and 3.6.2 that for each generator of type $a_{i,j}$ and $b_{i,j}$ in \mathcal{A}_Λ , the cellular \mathcal{A}_L has a pair of generators $a_{i,j}, \hat{a}_{i,j}$ and $b_{i,j}, \hat{b}_{i,j}$. A quick check shows $|\hat{x}_{i,j}| = |x_{i,j}| + 1$ for either type of generator. (If $A = (a_{i,j})$ sits above the 0-cell $\{x_1 = 0\}$ there are no new generators of type $\hat{a}_{i,j}$.)

Let $\hat{\partial}$ denote the suspension of $\partial := \partial_\Lambda$, which a priori may not equal ∂_L . We first verify that $\Gamma\partial = \hat{\partial}\Gamma$. This holds by definition in case (1) and by a tautology in case (2), so consider case (3). Recall $A = (a_{i,j})$ is the matrix of elements sitting over the 0-cell $\{x_1 = 0\}$.

$$\begin{aligned}\Gamma\partial(A) &= \Gamma(A^2) = A\Gamma(A) + \Gamma(A)A = AU + UA, \\ \hat{\partial}\Gamma(A) &= \hat{\partial}(U).\end{aligned}$$

Recall $U = U_r U_{r-1} \cdots U_1$. Assume we have shown that $AU + UA = \hat{\partial}U$ if $r = 1$. Then the general case follows from induction:

$$\hat{\partial}(U_{r+1}U) = \hat{\partial}(U_{r+1})U + U_{r+1}\hat{\partial}(U) = AU_{r+1}U + U_{r+1}AU + U_{r+1}AU + U_{r+1}UA = A(U_{r+1}U) + (U_{r+1}U)A.$$

Since $AI + IA = 0$ and $\hat{\partial}I = 0$, from (30) it suffices to show

$$\sum_{\ell < k} AE_{\ell,k} a_{\ell,k+1} + \sum_{k+1 < \ell} AE_{k+1,\ell} a_{k,\ell} + \sum_{\ell < k} a_{\ell,k+1} E_{\ell,k} A + \sum_{k+1 < \ell} a_{k,\ell} E_{k+1,\ell} A = \sum_{\ell < k} \partial a_{\ell,k+1} E_{\ell,k} + \sum_{k+1 < \ell} \partial a_{k,\ell} E_{k+1,\ell}.$$

Recall $a_{k,k+1} = 0$. Let $(M)_{ij}$ denote the (i, j) -entry of the $n \times n$ matrix M and δ_j^i denote the Kronecker delta. We compute each entry of each term in each of the four left hand side summands.

$$\begin{aligned}(AE_{\ell,k})_{ij} a_{\ell,k+1} &= \delta_k^j a_{i,\ell} a_{\ell,k+1} \text{ where } 1 \leq i < \ell < k, \\ (AE_{k+1,\ell})_{ij} a_{k,\ell} &= \delta_\ell^j a_{i,k+1} a_{k,\ell} \text{ where } 1 \leq i < k, k+1 < \ell \leq n, \\ a_{\ell,k+1} (E_{\ell,k} A)_{ij} &= \delta_\ell^i a_{\ell,k+1} a_{k,j} \text{ where } 1 \leq \ell < k, k+1 < j \leq n, \\ a_{k,\ell} (E_{k+1,\ell} A)_{ij} &= \delta_{k+1}^i a_{k,\ell} a_{\ell,j} \text{ where } k+1 < \ell < j \leq n.\end{aligned}$$

So the second and third summands cancel, while the first and fourth match the first and second on the right hand side.

Let $X = (x_{i,j})$ and $\hat{X} = (\hat{x}_{i,j})$ denote the strictly upper-triangular matrices of generators as in Section 3.6, modified appropriately if a crossing or cusp pair is involved. Then $\partial_L A$ is given by (2), $\partial_L B, \partial_L \hat{A}$ are given by (3), and $\partial_L \hat{B}$ is given by (4). We verify the identity for $\partial_L \hat{B}$, as $\partial_L \hat{A}$ is easier and $\partial_L A, \partial_L B$ are immediate.

Let $0 \geq t_1 > \dots > t_M \in \mathbb{R}_{x_1 < 0}^1 = \mathbb{R}_{r > 0}^1$ denote the 0-cells used to define the cellular DGA of Λ . We use polar coordinates $(r, \theta) \in \mathbb{R}_+^1 \times [-\pi, \pi]$ for \mathbb{R}_{x_1, x_2}^2 to refer to the 1-cells (polar lines) and 2-cells (polar rectangles) which define the cellular DGA of L . For example, all 1-cells that contain cusp or crossing loci are on $\{r = \text{const}\}$ lines. If a 1-cell is indicated to lie on $\{\theta = +\pi\}$ (resp. $\{\theta = -\pi\}$), this means that we consider the 1-cell as the boundary of a 2-cell lying counter-clockwise (resp. clockwise) to it. We orient all 1-cells in the positive polar coordinate direction. Fix a 2-cell $\{t_m \leq r \leq t_{m+1}\}$ with initial and terminal vertices $v_m = (t_m, -\pi)$, $v_{m+1} = (t_{m+1}, \pi)$. We use below that $B_{\{\theta = -\pi\}} = B_{\{\theta = \pi\}}$, $\hat{B} = \Gamma(B_{\theta = \pm\pi})$ and $\hat{A}_{v_l} = B_{\{r = t_l\}}$. Also, for the relative suspension case (so $t_1 = 0$), $B_{\{r = t_1\}} = \Gamma(A_{v_1})$ equals the 0 matrix when there are no cone points in case (2), and equals $U - I$ when there are in case (3).

$$\begin{aligned}\partial_L \hat{B} &= A_{v_{m+1}} \hat{B} + \hat{B} A_{v_m} + (I + B_{\{r = t_{m+1}\}})(I + B_{\{\theta = -\pi\}}) + (I + B_{\{\theta = \pi\}})(I + B_{\{r = t_m\}}) \\ &= A_{v_{m+1}} \hat{B} + \hat{B} A_{v_m} + B_{\{r = t_{m+1}\}} B_{\{\theta = -\pi\}} + B_{\{\theta = \pi\}} B_{\{r = t_m\}} + B_{\{r = t_{m+1}\}} + B_{\{r = t_m\}} \\ &= \Gamma(A_{v_{m+1}} B_{\{\theta = -\pi\}} + B_{\{\theta = \pi\}} A_{v_m} + A_{v_{m+1}} + A_{v_m}) \\ &= \Gamma(\partial_\Lambda B_{\{\theta = \pi\}}).\end{aligned}$$

□

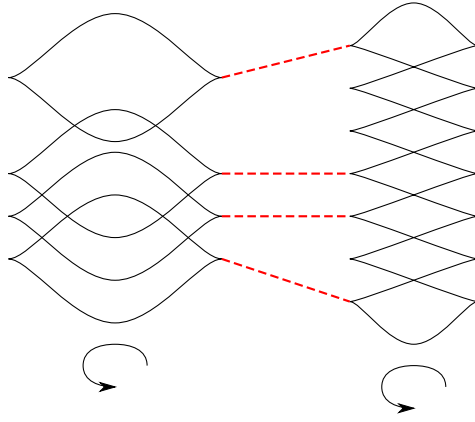


FIGURE 16. Construction of $L_{n,\sigma,\mathbf{a}}$ when $n = 4$, $\sigma(1) = 1$, $\sigma(2) = 3$, $\sigma(3) = 2$, $\sigma(4) = 6$, $\sigma(5) = 4$, $\sigma(6) = 7$, $\sigma(7) = 5$, $\sigma(8) = 8$ and $\mathbf{a} = (a_1, a_2, a_3, a_4) = (6, 3, 2, 0)$. The dotted red lines, are tubes with cusp edges.

5.5.4. *A comparison with older front-spun computations.* The first case (where there are no Legendrian arcs) is already known in the literature. The second two cases (Legendrian arcs spinning to possibly induce cone points, or not) are new.

Corollary 5.1. *The cellular DGA computation in the first example agrees with pre-existing computations of the LCH DGA, up to stable tame isomorphism.*

Proof. A partial computation of the LCH DGA of this example was done, up to certain quadratic terms, in [15, Proposition 4.17]. The complete LCH computation was done in [19, Theorem 1.1], which states that up to stable tame isomorphism, the LCH DGA of L is the suspension of the LCH DGA of Λ . The result then follows from Lemma 5.1 and Propositions 5.1 and 5.2. \square

5.6. **A family of examples.** The following generalizes a family of Legendrian spheres considered in [3].

We introduce Legendrians, $L_{n,\sigma,\mathbf{a}}$, where $n \geq 1$,

- $\sigma : \{1, 2, \dots, 2n\} \rightarrow \{1, 2, \dots, 2n\}$ is a permutation with the property that

$$i < j \text{ and } \sigma(i) > \sigma(j) \quad \Rightarrow \quad i \text{ even and } j \text{ odd},$$

- and $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{Z}^n$ satisfies $a_1 > a_2 > \dots > a_n = 0$.

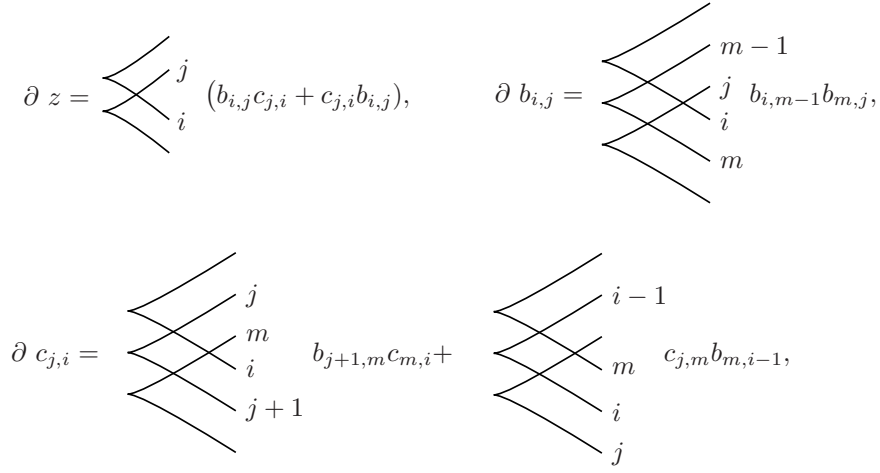
To construct $L_{n,\sigma,\mathbf{a}}$, start with n copies of the standard 1-dimensional Legendrian placed vertically above one another so that their left and right cusps all share common x -coordinates. For some pairs of unknots, the lower strand of the upper unknot crosses the upper strand of the lower unknot twice to form a Hopf link. This is specified by the permutation σ as follows: Labeling the strands near left cusps in with descending z -coordinate as $1, 2, \dots, 2n$, for $i < j$ the i -th strand crosses the j -th strand twice if and only if $\sigma(i) > \sigma(j)$. Arrange that the front diagram is symmetric, so that when we spin the front diagram around the midpoint (in the x -direction) it produces a stack of two dimensional unknots with one circular crossing arc for each $i < j$ with $\sigma(i) > \sigma(j)$. Call this spun front L_1 .

Next, to the right of the previously constructed front, we create one more Legendrian sphere, L_2 . Begin with a single 1-dimensional unknot; apply a_1 consecutive Type 1 Reidemeister moves to produce a front with $a_1 + 1$ left cusps and a_1 crossings. Then, spin around the midpoint to produce a 2-dimensional Legendrian sphere that is $a_1 + 1$ standard 2-dimensional unknots joined together by a sequence of a_1 cone points.

Finally, to arrive at $L_{n,\sigma,\mathbf{a}}$, we connect these two pieces by joining the i -th unknot from L_1 to the a_i -th cusp edge from L_2 by a tube whose cross-sections are Legendrian unknots. Here, we label the unknots of L_1 with descending z -coordinate, but label the cusp edges of L_2 with ascending z -coordinate, starting with 0. See Figure 16.

Theorem 5.4. *The DGA of $L_{n,\sigma,\mathbf{a}}$ is equivalent to a DGA with generators*

$$z, b_{i,j}, c_{\sigma(j),\sigma(i)}, \quad \text{for } i < j \text{ with } \sigma(i) > \sigma(j)$$

FIGURE 17. The differential for $L_{n,\sigma,\mathbf{a}}$.

and differentials

$$\begin{aligned}
\partial z &= \sum_{\substack{i < j \\ \sigma(i) > \sigma(j)}} (b_{i,j}c_{\sigma(j),\sigma(i)} + c_{\sigma(j),\sigma(i)}b_{i,j}) \\
\partial b_{i,j} &= \sum_{\substack{i < m < j \\ \sigma(i) > \sigma(m-1) \\ \sigma(m) > \sigma(j)}} b_{i,m-1}b_{m,j} \\
\partial c_{\sigma(j),\sigma(i)} &= \sum_{\substack{i < j < m \\ \sigma(j) < \sigma(m) < \sigma(i)}} b_{j+1,m}c_{\sigma(m),\sigma(i)} + \sum_{\substack{m < i < j \\ \sigma(j) < \sigma(m) < \sigma(i)}} c_{\sigma(j),\sigma(m)}b_{m,i-1}.
\end{aligned}$$

The grading is given by

$$|z| = 2, \quad |b_{i,j}| = 2(a_k - a_l) - 1, \quad |c_{\sigma(j),\sigma(i)}| = 2(a_l - a_k) + 2$$

where $i = 2k$ and $j = 2l - 1$.

The crossing configurations that lead to terms in ∂ are depicted in Figure 17. We prove Theorem 5.4 at the end of this subsection.

Among the $L_{n,\sigma,\mathbf{a}}$, there are many examples of pairs of Legendrians whose linearized homology groups have identical ranks, but are distinguished by the ring structure on linearized cohomology. We briefly recall the definition of the ring structure. Let $\epsilon : (\mathcal{A}, \partial) \rightarrow (\mathbb{Z}/2, 0)$ be a graded augmentation, and write $\mathcal{A} = \bigoplus_{n=0}^{\infty} V^{\otimes n}$ where V denotes the linear $\mathbb{Z}/2$ -span of the generators of \mathcal{A} . The conjugated differential

$$\partial^\epsilon = \Phi_\epsilon \partial (\Phi_\epsilon)^{-1}$$

has the form

$$\partial^\epsilon|_V = d_1 + d_2 + d_3 + \cdots, \quad \text{with } d_n : V \rightarrow V^{\otimes n}.$$

Dualizing gives operations

$$m_n : (V^*)^{\otimes n} \rightarrow V^*.$$

The linearized cohomology complex is (V^*, m_1) , and the operation m_2 descends to a well defined associative product of degree +1 on the linearized cohomology

$$LCH^*(L, \epsilon) := H^*(V^*, m_1).$$

The set of isomorphism types of the rings

$$\{LCH^*(L, \epsilon) \mid \epsilon \text{ any graded augmentation of } \mathcal{A}\}$$

depends only on the stable tame isomorphism type of (\mathcal{A}, ∂) and is hence a Legendrian isotopy invariant [6].

Observe that for any $L_{n,\sigma,\mathbf{a}}$ there is at least one augmentation, ϵ_0 , such that $m_1 = 0$. (Take ϵ_0 to vanish identically on generators.) Moreover, for $L_{n,\sigma,\mathbf{a}}$, the operation m_2 is independent of ϵ (due to the absence of monomials of degree 3 or more in ∂). Thus, the ring structure on $LCH^*(L, \epsilon_0)$ is independent of the choice of ϵ_0 (with $m_1 = 0$). This ring $LCH^*(L_{n,\sigma,\mathbf{a}}, \epsilon_0)$ is an invariant of $L_{n,\sigma,\mathbf{a}}$.

For example, consider the case where $\sigma = \sigma_0$ is

$$\sigma_0(1) = 1; \quad \sigma_0(2k) = 2k + 1; \quad \sigma_0(2k + 1) = 2k; \quad 1 \leq k \leq n - 1, \quad \sigma_0(2n) = 2n.$$

the Poincare polynomial of $LCH^*(L_{n,\sigma_0,\mathbf{a}}, \epsilon_0)$ is

$$P(t) = \sum_{k \in \mathbb{Z}} (\dim LCH^k(L_{n,\sigma_0,\mathbf{a}}, \epsilon_0)) t^k = t^2 + \sum_{l=1}^{n-1} \left(t^{2(a_l - a_{l+1}) - 1} + t^{2(a_{l+1} - a_l) + 2} \right).$$

By choosing the sequence $\mathbf{a} = (a_1, a_2, \dots, a_n)$ appropriately, this can be an arbitrary polynomial of the form

$$(34) \quad P(t) = t^2 + \sum_{l=1}^N (t^{2c_l - 1} + t^{2 - 2c_l})$$

for any $c_1, \dots, c_N > 0$.

Proposition 5.7. (1) *For arbitrary $L_{n,\sigma,\mathbf{a}}$, there exists n' and \mathbf{a}' such that the groups*

$$LCH^*(L_{n,\sigma,\mathbf{a}}, \epsilon_0) \quad \text{and} \quad LCH^*(L_{n',\sigma_0,\mathbf{a}'}, \epsilon_0)$$

have the same Poincare polynomial.

(2) *If there exists $i < j$ with $|i - j| \geq 2$ such that $\sigma(i) > \sigma(j)$, then these groups are not isomorphic as rings. In particular, $L_{n,\sigma,\mathbf{a}}$ and $L_{n',\sigma_0,\mathbf{a}'}$ are not Legendrian isotopic.*

Recall that for all 2-dimensional Legendrian spheres in standard contact \mathbb{R}^5 , such as $L_{n,\sigma,\mathbf{a}}$, the “classical” Legendrian invariants do not contain any information: there is only one smooth isotopy class, the Thurston-Bennequin number is 1, and the rotation class vanishes [15].

Proof. (1) follows since for arbitrary $L_{n,\sigma,\mathbf{a}}$ the Poincare polynomial has the form given in (34).

(2) follows since in $LCH^*(L_{n',\sigma_0,\mathbf{a}'}, \epsilon_0)$, the image of m_2 is 1-dimensional. (It is the span of z^* .) However, when $|i - j| \geq 2$ has $i < j$ and $\sigma(i) > \sigma(j)$, there is some $i < m - 1 < m < j$ such that $\sigma(m - 1) < \sigma(j) < \sigma(i) < \sigma(m)$. This implies that $m_2(b_{i,m-1}^*, b_{m,j}^*) = b_{i,j}^*$, so that the image of m_2 has dimension 2 or more. \square

As a specific example where (2) applies, consider

$$\begin{aligned} n &= 3, & \sigma &= (2 \ 4 \ 5 \ 3), & \text{and } \mathbf{a} &= (2, 1, 0); \\ n' &= 4, & \sigma_0 &= (2 \ 3)(4 \ 5)(6 \ 7), & \text{and } \mathbf{a}' &= (4, 2, 1, 0). \end{aligned}$$

Remark 5.5. In the case of $\sigma = \sigma_0$ and $n = 2$, the Legendrians $L_{2,\sigma_0,\mathbf{a}}$ are considered in [3, Section 6] with the generating family homology computed. After correcting for a small error in the computation in [3] of the Maslov potential, the generating family homology has the same Poincare polynomial as $LCH^*(L_{2,\sigma_0,\mathbf{a}}, \epsilon_0)$.

Proof of Theorem 5.4. We use the extended version of the cellular DGA from Propositions 5.5 and 5.6, so we work with a decomposition of $\pi_x(L_{n,\sigma,\mathbf{a}})$ where all of the crossing arcs (resp. cone points) of L_1 (resp. L_2) correspond to a single circle (resp. single 0-cell) in the 1-skeleton of \mathcal{E} . Moreover, by methods similar to those used in the proof of Proposition 5.5, we can position the cusp arcs so that together their projection is as in Figure 18. In more detail, the cusp edges of the L_2 part of $L_{n,\sigma,\mathbf{a}}$ all project to a circle in \mathbb{R}^2 ; the cusp edges that are part of $L_{n,\sigma,\mathbf{a}}$ to the left of L_2 (i.e., L_1 together with the tubes connecting L_1 to L_2) project to an arc with two end points on the L_2 circle. This leads to the cellular decomposition pictured in Figure 18. Since the cusp edges above any given edge all involve distinct pairs of sheets, the only modification to the usual definition of the DGA is that when computing the differential of bordering cells, at 0-cells (resp. 1-cells) we insert one $N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ block (resp. 2×2 0 block) for every pair of sheets that cusp above the 0-cell. In particular, above the 0-cells v_0 and v_1 the associated matrices have N blocks all along the diagonal.

We DO NOT cancel $c_{2a_1+1,2a_1+2}^3$. Instead, set $z = c_{2a_1+1,2a_1+2}^3$, and observe that

$$(39) \quad \partial z \doteq (\widehat{B_3})_{1,2} + (\widehat{B_3})_{3,4} + \cdots + (\widehat{B_3})_{2a_1+1,2a_1+2}.$$

Finally, we cancel all remaining C_3 generators, with themselves; note that (36) gives for $i < 2l < 2l+1 < j$,

$$\partial \boxed{c_{i,2l+1}^3} = \boxed{c_{i,2l}^3} + X; \quad \partial \boxed{c_{2l,j}^3} = \boxed{c_{2l+1,j}^3} + Y$$

where X belongs to the sub-algebra generated by generators x satisfying $x \prec_{\mathcal{A}} c_{i,2l}^3$, and similar for Y .

To summarize, the only generator remaining from the matrices A_2, B_4, C_3 is $z = c_{2a_1+1,2a_1+2}^3$ whose differential is given by (39)

Step 2. *Canceling generators in the L_1 part of L .*

Above L_1 , L has $2n$ sheets. The matrices B_1, B_3, C_2, C_3 have generators in all of there entries with $1 \leq i < j \leq n$. The matrices A_1 and B_2 have generators $a_{i,j}^1$ and $b_{i,j}^2$ in the entries with $i < j$ and $\sigma(i) < \sigma(j)$, with the remaining entries 0 (as specified Proposition 5.5). There are no generators associated to the vertices v_0 and v_1 . As indicated in Figure 18, we label sheets above these cells as they appear above C_3 . The notation for these matrices is fixed. We have differentials

$$(40) \quad \partial B_1 = A_1(I + B_1) + (I + B_1)N,$$

$$(41) \quad \partial C_1 = Q_\sigma A_1 Q_\sigma^{-1} C_1 + C_1 Q_\sigma A_1 Q_\sigma^{-1} + Q_\sigma (I + B_2) Q_\sigma^{-1} + I,$$

$$(42) \quad \partial C_2 = N C_2 + C_2 N + (I + B_3) + (I + B_1)(I + B_2)(I + B_1)^{-1}.$$

Recall $Q_\sigma = \Sigma E_{\sigma(i),i}$ so that the (i,j) -entry of $Q_\sigma A_1 Q_\sigma^{-1}$ is $a_{\sigma^{-1}(i),\sigma^{-1}(j)}^1$.

Start by canceling

$$\partial \boxed{C_2} = \boxed{B_3} + \cdots.$$

Observe, that in the quotient

$$(43) \quad (B_3)_{i,i+1} \doteq (B_2)_{i,i+1}$$

(because the $(i, i+1)$ -entry of $N C_3$ and $C_3 N$ is 0 since both factors are upper triangular, and the $(i, i+1)$ -entry of $(I + B_1)(I + B_2)(I + B_1)^{-1}$ is the sum of the $(i, i+1)$ -entries of the three factors).

Next, we will cancel as many of the entries of B_1 with A_1 as we can. When doing so, in order to meet the hypothesis of Theorem 2.1, we will need to modify the ordering of generators. For the entries of B_1 and A_1 , we require that $x_{i',j'} \prec y_{i,j}$ if and only if $j' - i' < j - i$ (where each of x and y has the form a^1 or b^1). Remaining generators are all larger than those from B_1 and A_1 and are ordered as in Lemma 4.1.

With this ordering, we can cancel

$$\partial \boxed{b_{i,j}^1} = \boxed{a_{i,j}^1} + \cdots, \quad \forall i < j \text{ with } \sigma(i) < \sigma(j).$$

In the quotient, the values of $a_{i,j}^1$ are as follows, depending on the parity of i and j :

$$(44) \quad \begin{array}{ll} i \text{ odd}, j \text{ even:} & a_{i,j}^1 \doteq \begin{cases} 1, & (i,j) = (2l-1, 2l) \\ 0, & \text{else.} \end{cases} \\ i \text{ even}, j \text{ even:} & a_{i,j}^1 \doteq b_{i,j-1}^1 \\ i \text{ odd}, j \text{ odd:} & a_{i,j}^1 \doteq b_{i+1,j}^1 \end{array}$$

[These formula are verified using induction on $j - i$ and the relation

$$a_{i,j}^1 \doteq \sum_{i < m < j} a_{i,m}^1 b_{m,j}^1 + (B_1 N)_{i,j}.$$

In doing so, it is useful to note that if $b_{m,j}^1$ is not 0 in the quotient then, since $\sigma(m) > \sigma(j)$, we must have m even and j odd.]

The generators $b_{i,j}^1$ with $i < j$ and $\sigma(i) > \sigma(j)$ will not be canceled. They have differentials

$$(45) \quad \partial b_{i,j}^1 = \sum_{i < m < j} a_{i,m}^1 b_{m,j}^1 \doteq \sum_{\substack{i < m < j \\ \sigma(i) > \sigma(m-1) \\ \sigma(m) > \sigma(j)}} b_{i,m-1}^1 b_{m,j}^1.$$

[At the first equality, note that there is no $a_{i,j}^1$ generator, and for the second equality use that $b_{m,j}^1 = 0$ unless m is even.]

Finally, we cancel as many entries of C_1 as we can with the entries of B_2 , using

$$\partial \boxed{C_1} = Q_\sigma^{-1} \boxed{B_2} Q_\sigma + \dots$$

In doing so, it is convenient to rewrite (41) as

$$\partial Q_\sigma^{-1} C_1 Q_\sigma = A_1 Q_\sigma^{-1} C_1 Q_\sigma + Q_\sigma^{-1} C_1 Q_\sigma A_1 + B_2,$$

giving the entry-by-entry formula

$$\partial c_{\sigma(i),\sigma(j)}^1 = \sum_{i < m, \sigma(i) < \sigma(m) < \sigma(j)} a_{i,m}^1 c_{\sigma(m),\sigma(j)}^1 + \sum_{m < j, \sigma(i) < \sigma(m) < \sigma(j)} c_{\sigma(i),\sigma(m)}^1 a_{m,j}^1 + \begin{cases} b_{i,j}^2, & i < j \\ 0, & i > j. \end{cases}$$

Thus, for all $i < j$ with $\sigma(i) < \sigma(j)$, we cancel

$$\partial \boxed{c_{\sigma(i),\sigma(j)}^1} = \boxed{b_{i,j}^2} + \dots$$

[To apply Theorem 2.1, we order so that all entries of B_2 are greater than all entries of B_1 (the A_1 generators belong to the subalgebra generated by the B_1 generators in the current quotient), and so that $c_{\sigma(i),\sigma(j)}^1 < b_{i',j'}^2$ if and only if $\sigma(j) - \sigma(i) < j' - i'$.] All $c_{\sigma(i),\sigma(j)}^1$ generators with $j < i$ and $\sigma(i) < \sigma(j)$ remain in the quotient. For such generators, j is even and i is odd so we use (44) to compute

$$(46) \quad \begin{aligned} \partial c_{\sigma(i),\sigma(j)}^1 &= \sum_{i < m, \sigma(i) < \sigma(m) < \sigma(j)} a_{i,m}^1 c_{\sigma(m),\sigma(j)}^1 + \sum_{m < j, \sigma(i) < \sigma(m) < \sigma(j)} c_{\sigma(i),\sigma(m)}^1 a_{m,j}^1 \\ &\doteq \sum_{i < m; j < m; \sigma(i) < \sigma(m) < \sigma(j)} b_{i+1,m}^1 c_{\sigma(m),\sigma(j)}^1 + \sum_{m < j; m < i; \sigma(i) < \sigma(m) < \sigma(j)} c_{\sigma(i),\sigma(m)}^1 b_{m,j-1}^1. \end{aligned}$$

Note that these formulas agree with those from the statement of the theorem with the role of i and j interchanged.

To summarize, in Step 1 and Step 2 we have now cancelled all of the generators besides z and those $b_{i,j}^1$ and $c_{\sigma(j),\sigma(i)}^1$ with $i < j$ and $\sigma(i) > \sigma(j)$. The differentials of the $b_{i,j}^1$ and $c_{\sigma(j),\sigma(i)}^1$ have been computed in (45) and (46). It remains only to calculate ∂z . Using that

$$\begin{aligned} b_{2l-1,2l}^2 &\doteq \sum_{2l-1 < m, \sigma(2l-1) < \sigma(m) < \sigma(2l)} a_{2l-1,m}^1 c_{\sigma(m),\sigma(2l)}^1 + \sum_{m < 2l, \sigma(2l-1) < \sigma(m) < \sigma(2l)} c_{\sigma(2l-1),\sigma(m)}^1 a_{m,2l}^1 \\ &\doteq \sum_{2l < m, \sigma(m) < \sigma(2l)} b_{2l,m}^1 c_{\sigma(m),\sigma(2l)}^1 + \sum_{m < 2l-1, \sigma(2l-1) < \sigma(m)} c_{\sigma(2l-1),\sigma(m)}^1 b_{m,2l-1}^1. \end{aligned}$$

[At the second equality, we used that the condition on the first (resp. second) summation implies m crosses $2l$ (resp. m crosses $2l-1$), so m is odd (resp. even).] Since

$$\begin{aligned} \partial z &\doteq (\widehat{B_3})_{1,2} + (\widehat{B_3})_{3,4} + \dots + (\widehat{B_3})_{2a_1+1,2a_1+2} \\ &= b_{1,2}^3 + b_{3,4}^3 + \dots + b_{2l-1,2l}^3 \\ &\doteq \sum_{l=1}^n b_{2l-1,2l}^2 \end{aligned}$$

the result follows. [The second equality uses that $(\widehat{B_3})_{2r-1,2r}$ is 0 unless there is a tube connecting one of the unknots on the left to the r -th cusp edge. The final inequality uses (43).]



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